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A NEW APPROACH FOR SOLVING INITIAL AND BOUNDARY VALUE PROBLEMS OF BRATU-TYPE

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Abstract: One of the newest analytical methods to solve nonlinear equations is Variational Iteration Method-II which is an accurate and a rapid convergence method in finding the approximate solution for nonlinear equations. In these papers, VIM-II is implemented to solve the initial and boundary value problems of Bratu-type which are widely applicable in fuel ignition of the combustion theory and heat transfer. Several examples are given to confirm the efficiency and the accuracy of the proposed algorithm.
Keywords: Variational Iteration Method-II (VIM-II), Bratu's problem, boundary value problems, initial value problems

1. INTRODUCTION

It is well known that Bratu's boundary value problem in one-dimensional planar coordinates is of the form [1,2,3]

$$u''+\lambda e^{u} = 0, \quad 0 < x < 1, \tag{1}$$

with boundary conditions $u(0) = 0$, $u(1) = 0$

The standard Bratu type problem (1) was used to model a combustion problem in a numerical slab. The Bratu models appear in a number of applications such as the fuel ignition of the thermal combustion theory and in the Chandrasekhar model of the expansion of the universe [4]. It stimulates a thermal reaction process in a rigid material where the process depends on the balance between chemically generated heat and heat transfer by conduction.

In recent years, several such techniques have drawn special attention, such as inverse scattering method [5], Adomian's decomposition method ADM [6], the Variational Iteration Method [7-10], Homotopy Perturbation Method [11-14], Energy Balance Method [15], as well as Homotopy Analysis Method (HAM). After that, many types of nonlinear problems were solved by the HAM by others [16, 17]. One of the newest analytical methods to solve nonlinear equations is Reconstruction of variational Iteration Method (VIM-II) which is an accurate and a rapid convergence method in finding the approximate solution for nonlinear equations. By applying Laplace Transform, VIM-II overcomes the difficulty of the perturbation techniques and other variational methods in case of using small parameters and lagrange multipliers, respectively. Reducing the size of calculations and omitting the difficulty arising in calculation of nonlinear intricately terms are other advantages of this method. Recently, Noor and Mohyud-Din [18] employed homotopy perturbation method and the variational iteration decomposition method for finding the solution of these problems. Inspired and motivated by the ongoing research in this area, we applied a relatively new technique, the Reconstruction of Variational Iteration Method for solving initial and boundary value problems of Bratu type models.

The exact solution to (1) is given as: [1,2,4,19,20]

$$u(x) = 2\ln\left[\frac{\cosh((x-\frac{1}{2})\frac{\theta}{2})}{\cosh(\frac{\theta}{4})}\right],$$

where θ satisfies

$$\theta = \sqrt{2\lambda}\cosh(\frac{\theta}{4})$$

The Bratu problem has zero, one or two solutions when $\lambda > \lambda c$, $\lambda = \lambda c$ and $\lambda < \lambda c$, respectively, where the critical value λc satisfies the equation

$$1 = \frac{1}{4}\sqrt{2\lambda_c}\sinh(\frac{\theta_c}{2}).$$

It was evaluated in [1,2,4,19,20] that the critical value λc is given by

$$\lambda_c = 3.51383071\quad 9$$

The basic motivation of the present work is to introduce a reliable treatment of two boundary value problems of Bratu-type model, given by

$$u''-\pi^2 e^u = 0, \quad 0 < x < 1,$$

with boundary conditions *u(0) = 0, u(1) = 0* and an initial value problem of the Bratu-type

$u''-2e^{-u} = 0, \quad 0 < x < 1,$ with initial conditions *u(0) = 0, u(1) = 0*

2. ANALYSIS OF THE VIM-II

To To clarify the basic ideas of our proposed method in [21], we consider the following differential equation same as VIM based on Lagrange multiplier [22]:

$$Lu(x_1,\cdots,x_k) + Nu(x_1,\cdots,x_k) = f(x_1,\cdots,x_k) \tag{2}$$

By suppose that

$$Lu(x_1,\cdots,x_k) = \sum_{i=0}^{k} L_{xi}u(x_i) \tag{3}$$

Where L is a linear operator, N a nonlinear operator and $f(x_1,\cdots,x_k)$ an inhomogeneous term. we can rewrite equation (2) down a correction functional as follows:

$$L_{xj}u(x_j) = \underbrace{f(x_1,\cdots,x_k) - Nu(x_1,\cdots,x_k) - \sum_{\substack{i=0\\ i\neq j}}^{k} L_{xi}u(x_i)}_{h((x_1,\cdots,x_k),u(x_1,\cdots,x_k))} \tag{4}$$

therefore

$$L_{xj}u(x_j) = h((x_1,\cdots,x_k),u(x_1,\cdots,x_k)) \tag{5}$$

with artificial initial conditions being zero regarding the independent variable x_j.

By taking Laplace transform of both sides of the equation (5) in the usual way and using the artificial initial conditions, we obtain the result as follows

$$P(s).U(x_1,\cdots,x_{i-1},s,x_{i+1},x_k) = H((x_1,\cdots,x_{i-1},s,x_{i+1},x_k),u) \tag{6}$$

Where P(s) is a polynomial with the degree of the highest derivative in equation (6), (the same as the highest order of the linear operator L_{xi}). The following relations are possible;

$$\ell[\mathrm{h}] = \mathrm{H} \tag{7-a}$$

$$\mathrm{B(s)} = \frac{1}{\mathrm{P(s)}} \tag{7-b}$$

$$\ell[\mathrm{b}(\mathrm{x_i})] = \mathrm{B(s)} \tag{7-c}$$

Which that in equation (7-a) the function $H((x_1,\cdots,x_{i-1},s,x_{i+1},x_k),u)$ and $h((x_1,\cdots,x_{i-1},x_i,x_{i+1},x_k),u)$

have been abbreviated as H, h respectively.

Hence, rewrite the equation (6) as;

$$U(x_1,\cdots,x_{i-1},s,x_{i+1},x_k)=H((x_1,\cdots,x_{i-1},s,x_{i+1},x_k),u).B(s) \tag{8}$$

Now, by applying the inverse Laplace Transform on both sides of equation (8) and by using the (7-a) - (7-c), we have;

$$\mathrm{u}(\mathrm{x}_1,\cdots,\mathrm{x}_{\mathrm{i-1}},\mathrm{x}_\mathrm{i},\mathrm{x}_{\mathrm{i+1}},\mathrm{x}_\mathrm{k})=\int_0^{\mathrm{x}_\mathrm{i}}\mathrm{h}((x_1,\cdots,x_{i-1},\tau,x_{i+1},x_k),u).b(x_i-\tau)d\tau \tag{9}$$

Now, we must impose the actual initial conditions to obtain the solution of the equation (2). Thus, we have the following iteration formulation:

$$\begin{aligned}\mathrm{u}_{\mathrm{n+1}}(\mathrm{x}_1,\cdots,\mathrm{x}_{\mathrm{i-1}},\mathrm{x}_\mathrm{i},\mathrm{x}_{\mathrm{i+1}},\mathrm{x}_\mathrm{k})&=\mathrm{u}_0(\mathrm{x}_1,\cdots,\mathrm{x}_{\mathrm{i-1}},\mathrm{x}_\mathrm{i},\mathrm{x}_{\mathrm{i+1}},\mathrm{x}_\mathrm{k})\\&+\int_0^{\mathrm{x}_\mathrm{i}}\mathrm{h}((x_1,\cdots,x_{i-1},\tau,x_{i+1},x_k),u).b(x_i-\tau)d\tau\end{aligned} \tag{10}$$

where u_0 is initial solution with or without unknown parameters. Assuming u_0 is the solution of ¸ Lu, with initial/boundary conditions of the main problem, In case of no unknown parameters, u_0 should satisfy initial/ boundary conditions. When some unknown parameters are involved in u_0, the unknown parameters can be identified by initial/boundary conditions after few iterations, this technology is very effective in dealing with boundary problems. It is worth mentioning that, in fact, the Lagrange multiplier in the He's variational iteration method is $\lambda(\tau)=b(x_i-\tau)$ as shown in [21].

The initial values are usually used for selecting the zeroth approximation u_0. With u_0 determined, then several approximations u_n $n>0$, follow immediately. Consequently, the exact solution may be obtained by using

$$\mathrm{u}(\mathrm{x}_1,\cdots,\mathrm{x}_{\mathrm{i-1}},\mathrm{x}_\mathrm{i},\mathrm{x}_{\mathrm{i+1}},\mathrm{x}_\mathrm{k})=\lim_{\mathrm{n}\to\infty}\mathrm{u}_\mathrm{n}(\mathrm{x}_1,\cdots,\mathrm{x}_{\mathrm{i-1}},\mathrm{x}_\mathrm{i},\mathrm{x}_{\mathrm{i+1}},\mathrm{x}_\mathrm{k}). \tag{11}$$

3. APPLICATION OF THE VARIATIONAL ITERATION METHOD-II

In order to assess the advantages and the accuracy of VIM-II for solving initial and boundary value problems of the Bratu-type, The fact that the suggested technique solves nonlinear problems without using Adomian polynomials can be considered as a clear advantage of this method over the decomposition method. Moreover, we have also considered an example of higher dimensional initial boundary value problem [22]. For the sake of comparison we take the same examples as considered by Wazwaz [18], Noor and Mohyud-Din [20].

Example1.

To apply the VIM-II, first we Consider the following Bratu-type model

$$u''-\pi^2e^u=0,\quad 0<x<1, \tag{12}$$

With boundary condition:

$$u(0)=0,\ u(1)=0. \tag{13}$$

At first rewrite eq. (1) based on selective linear operator as

$$\ell\{u(x)\}=u''=\overbrace{\pi^2e^u}^{h(x,u)} \tag{14}$$

Now by applying the Laplace Transform on both sides of Eq. (14) by using the artificial initial condition [21]:

$$u(x)=\int_0^x(\varepsilon-x)h(\varepsilon,u)d\varepsilon \tag{15}$$

Therefore, by using the Eq. (9), (10) one can obtain the following VIM-II's iteration formula

$$u_{n+1}=u_0+\int_0^x(\varepsilon-x)h(\varepsilon,u)d\varepsilon \tag{16}$$

Now we start with an arbitrary initial approximation $u_0 = \alpha x$, that satisfies the initial condition and by using the VIM-II iteration formula (16), we have the following successive approximation

$$u_1(x) = ax - \frac{\pi^2}{a^2} e^{-ax} + ax + 1, \tag{17}$$

$$u_2(x) = ax - \frac{\pi^2}{a^2} e^{-ax} + ax + 1 - \frac{\pi^4}{4a^4} e^{-2ax} + 4axe^{-ax} - 4e^{-ax} + 2ax + 5 \tag{18}$$

The series solution is given by

$$u(x) = ax - \frac{\pi^2}{a^2} e^{ax} + ax + 1 - \frac{\pi^4}{4a^4} e^{2ax} + 4axe^{ax} + 4e^{ax} + 2ax + 5 + \frac{\pi^6}{12a^6} e^{3ax} + 6e^{2ax}$$
$$+1 + ax + 3e^{ax} 2a^2 x^2 + 6ax + 5 + 6ax - 22 + \ldots.. \tag{19}$$

or equivalently

$$u(x) = ax + \frac{\pi^2}{2!} x^2 + \frac{\pi^2 a}{3!} x^3 + \left(\frac{\pi^2 a^2 + \pi^4}{4!} \right) x^4 + \left(\frac{\pi^2 a^3 + 4\pi^4 a}{5!} \right) x^5 + \left(\frac{11\pi^4 a^2 + \pi^2 a^4 + 4\pi^6}{6!} \right) x^6$$
$$+ \left(\frac{26\pi^4 a^3 + \pi^2 a^5 + 34\pi^6 a}{6!} \right) x^7 + \ldots \tag{20}$$

Imposing the boundary condition $u(1) = 0$ leads to obtain $a = \pi$ and consequently, the closed form solution is given as,

$$u(x) = -\ln\left(1 + \cos\left((\frac{1}{2} + x)\pi \right) \right). \tag{21}$$

Example 2. Consider the following initial value problem of the Bratu-type

$$u'' - 2e^u = 0, \qquad 0 < x < 1 \tag{22}$$

Subject to the initial conditions:

$$u(0) = 0, \qquad\qquad u(1) = 0 \tag{23}$$

Such as previous examples, for implementation of the VIM-II technique, first of all we need to choose the auxiliary linear operator as

$$\ell\{u(x)\} = u'' = \overbrace{2e^u}^{h(x,u)} \tag{24}$$

Accordingly, after taking Laplace Transform by using the artificial initial condition as in [21], on both side of Eq. (24), the following VIM-II iteration formula can be obtained

$$u_{n+1} = u_0 + \int_0^x (\varepsilon - x) h(\varepsilon, u) d\varepsilon \tag{25}$$

By the VIM-II'S recurrent formula in Eq. (25), the terms of the sequence $\{u_n\}$ are constructed as follows, so that we choose its initial approximate solution as $u_0(x) = 0$.

$$u_1(x) = x^2$$
$$u_2(x) = x^2 + \frac{1}{6}x^4,$$
$$u_3(x) = x^2 + \frac{1}{6}x^4 + \frac{2}{45}x^6,$$
$$u_4(x) = x^2 + \frac{1}{6}x^4 + \frac{2}{45}x^6 + \frac{17}{1260}x^8,$$
$$u_5(x) = x^2 + \frac{1}{6}x^4 + \frac{2}{45}x^6 + \frac{17}{1260}x^8 + \frac{62}{14175}x^{10}$$
$$u_6(x) = x^2 + \frac{1}{6}x^4 + \frac{2}{45}x^6 + \frac{17}{1260}x^8 + \frac{62}{14175}x^{10} + \frac{691}{467775}x^{12},$$

The series solution is given by

$$u(x) = x^2 + \frac{1}{6}x^4 + \frac{2}{45}x^6 + \frac{17}{1260}x^8 + \frac{31}{14175}x^{10} + \frac{691}{467775}x^{12} + ..., \tag{26}$$

or equivalently

$$u(x) = -2\left(-\frac{1}{2}x^2 - \frac{1}{12}x^4 - \frac{1}{45}x^6 - \frac{17}{2520}x^8 - \frac{31}{14175}x^{10} - \frac{691}{935550}x^{12} + ...\right). \tag{27}$$

The exact solution is given by

$$u(x) = -2\ln\cos(x) \tag{28}$$

4. CONCLUSION

In this paper, we successfully apply Variational Iteration Method-II (VIM-II) for finding the exact solution of Bratu-type equation. Simplicity and requiring less computation, rapid convergence, and high accuracy are advantages of this technique. Moreover, the VIM-II reduces the size of calculations by not requiring the tedious Adomian polynomials, and hence the iteration is direct and straightforward. The results reported here provide further evidence of the usefulness of VIM-II for finding the analytic and numeric solutions for the linear and nonlinear diffusion equations and, it is also a promising method to solve different types of nonlinear equations in mathematical physics.

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