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ON THE INTEGRAL EQUATIONS OF VOLTERRA CONCERNING THE LARGE TIME SCALE

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Abstract: In this paper the authors introduce a sum of singular kernels to well describe the experimental data concerning the creep and stress relaxation of viscoelastic solids in large time interval. The solution (concerning the creep) of the respective integral equations describing the stress relaxation is obtained with the help of resolving kernel represented as a sum of the respective resolving kernels concerning each sum member. Experimental results for polyisoprene rubber illustrate the applicability of the proposed kernels sum.
Keywords: viscoelasticity, hereditary theory, singular kernel

1. INTRODUCTION

It is well known that the Boltzmann hereditary theory using integral equations of Volterra [1] can well describe the creep and stress relaxation of different viscoelastic solids. Let introduce the following integral equations to describe the mechanical behavior of such as solid [1,2]

$$\sigma(t) = E\varepsilon(\mathrm{t}) - E\int_0^t R(t,\tau)\varepsilon(\tau)d\tau\,, \tag{1}$$

Here $\sigma(t)$ is the stress as a function of the time t, $\varepsilon(t)$ is the imposed strain, $E$is the Young module and $R(t,\tau)$ is the relaxation kernel which can be found from stress relaxation tests. The solution of equation (1) introducing Neumann series to find the resolving kernel $K(t,\tau)$ is [3]

$$\varepsilon(t) = \frac{1}{E}\sigma(t) + \frac{1}{E}\int_0^t K(t,\tau)\sigma(\tau))d\tau\,, \tag{2}$$

The above mentioned integral equations of Volterra has been longtime employed to describe the viscoelastic behavior of polymers, rubbers and other materials [1, 2]. Due to the extremely high strain (stress) rate at the beginning in creep (relaxation) conditions one needs to introduce singular kernels. These kernels well describe the viscoelastic behavior, but in small time region. In this paper we will employ the following kernels [3]

$$R(t) = A\frac{e^{\beta t}}{t^{\alpha}}\,, \tag{3}$$

whose resolving kernel (the creep kernel) looks like [3]

$$K(t) = \frac{e^{\beta t}}{t}\sum_{n=1}^{\infty} A\Gamma(\alpha)^n t^{\alpha n} / \Gamma(\alpha n)\,. \tag{4}$$

Here $\Gamma(\alpha)$ is the gamma function.

It is very difficult to fit the parameters in equation (4) in order to well describe in large time interval the relaxation (or creep) experimental data using only one kernel. This situation corresponds with one elementary viscoelastic model and many researchers introduce a great number of elementary models [1] to better describe theviscoelastic behavior. In other words if we well fit the experimental parameters A, α, β and thus our stress relaxation curve well describe the experimental points at the beginning, this curve do not coincides with the experimental data in the end of the stress relaxation time interval i.e. they fails concerning the great time values.

2. GENERAL FRAMEWORK

In order to increase the creep or stress relaxation time interval and thus well describe the experimental data from the beginning to the end in the case of large time interval, we propose to involve a sum of singular kernels as follows:

$$R(t) = \sum_{n=1}^{N} \hat{R}_i(t) \ , \tag{5}$$

with $\hat{R}_i(t) = A_i \dfrac{e^{\beta_i t}}{t^{\alpha_i}}$.

In this case we will prove that the solution has the form

$$\varepsilon(t) = \frac{1}{E}\sigma(t) \ + \frac{1}{E}\int_0^t \sum_{i=1}^{N} \hat{K}_i(t,\tau)\sigma(\tau)d\tau \ , \tag{6}$$

with
$$\hat{K}_i(t) = \frac{e^{\beta_i t}}{t}\sum_{n=1}^{\infty} A_i \Gamma(\alpha_i)^n t^{\alpha_i n} / \Gamma(\alpha_i n) \tag{6a}$$

Using the Neumann successive approximations introduced as in [5] to the resolving kernelin equation (2) we have

$$K(t) = \sum_{m=1}^{\infty} R_m(t) \ , \tag{7}$$

where the m-*th* approximation is represented as

$$R_m(t) = \int_0^t R_1(t,x) R_{m\ 1}(x,\tau)dx \ \text{ and } \ R_0(t) = R_1 \, . \tag{8}$$

If the first equation (5) is valid we can write

$$K_m(t) = \int_0^t \sum_{i=1}^{N} R_1(t,x) R_{m\ 1}(x,\tau)dx \, , \tag{9}$$

The resolving kernel taking into account equations (7, 8) can be represented as

$$\begin{aligned}\hat{K}(t,\tau) &= \sum_{m=1}^{\infty} \int_0^t \sum_{i=1}^{N} \hat{R}_i(t,x) R_{m\ 1}(x,\tau)dx = \sum_{m=1}^{\infty} \int_0^t \hat{R}_1(t,x) R_{m\ 1}(x,\tau)dx + \\ &\sum_{m=1}^{\infty} \int_0^t \hat{R}_2(t,x) R_{m\ 1}(x,\tau)dx + \ldots + \sum_{m=1}^{\infty} \int_0^t \hat{R}_N(t,x) R_{m\ 1}(x \quad \tau)dx \\ &= \sum_{i=1}^{N} \sum_{m=1}^{\infty} \hat{R}_{m,i}(t)) = \sum_{i=1}^{N} \hat{K}_i(t)\end{aligned} \tag{10}$$

Thus, to the solution of the linear equation (1) with kernel (5) we have

$$\varepsilon(t) = \frac{1}{E}\sigma(t) \ + \frac{1}{E}\int_0^t \sum_{i=1}^{N} \hat{K}_i(t,\tau)\sigma(\tau)d\tau \ , \tag{11}$$

This approach can be generalized in the case of nonlinear elastoviscous behavior. Assuming similarity of the isochrones stress relaxation curves one can apply the following nonlinear integral equation [4]

$$\sigma(t) = \ (\varepsilon(t)) - E\int_0^t \hat{R}(t,\tau) \ (\varepsilon(\tau))d\tau \, . \tag{12}$$

Here $\varphi(\varepsilon(\mathrm{t}))$ is the instantaneous stress-strain curve. To well describe this curve one can apply the Ogden relation [2]

$$\varphi(\varepsilon)=\mu_1(\lambda(\varepsilon)^{1.5\alpha_1}-1)/\lambda(\varepsilon)^{0.5\alpha_1}+\mu_2(\lambda(\varepsilon)^{1.5\alpha_2}-1)/\lambda(\varepsilon)^{0.5\alpha_2}+\mu_3(\lambda(\varepsilon)^{1.5\alpha_3}-1)/\lambda(\varepsilon)^{0.5\alpha_3} \quad . \tag{13}$$

Here $\mu_1,\mu_2,\mu_3,\alpha_1,\alpha_2,\alpha_3$ are parameters obtained from instantaneous stress-strain tests. The solution of equation (12) can be represented as follows:

$$\varphi(\varepsilon(t))=\sigma(t)+\int_0^t \hat{K}(t,\tau)\sigma(\tau))d\tau \, . \tag{14}$$

To obtain the strain curve (nonlinear creep) one should use the inverse function $\psi(\varepsilon(t))=\varphi^{-1}(\varepsilon(t))$. In equations (13, 14) the kernels $\hat{R}(t,\tau)$ and $\hat{K}(t,\tau)$ are the same as in equations (5, 10) and can be identified from small strain tests.

3. EXPERIMENTAL RESULTS AND COMPARISONS

Here we will illustrate the applicability of our approach concerning the stress relaxation and the respective creep curves for polyisoprene rubber at large time interval $0\le t\le 8000$ seconds.The Ogden and the stress relaxation kernel parameters are identified from figures 1 and 2. The creep curves (small strains) are illustrated in figures 3. In figure 4 one can see the creep curves in the case of large strains – equation (14).

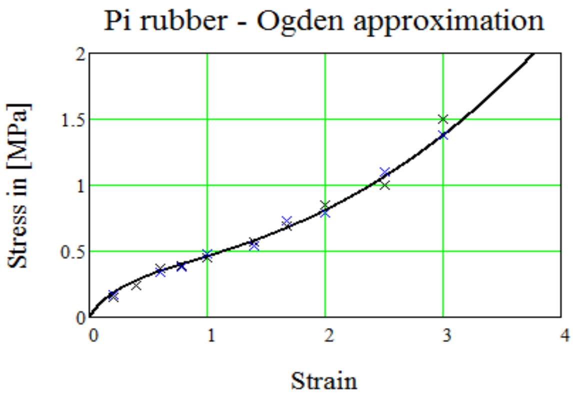

Figure 1. Instantaneous stress-strain curves

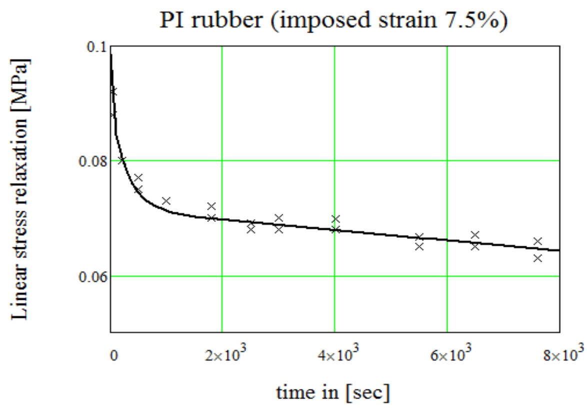

Figure 2. Linear stress relaxation curves

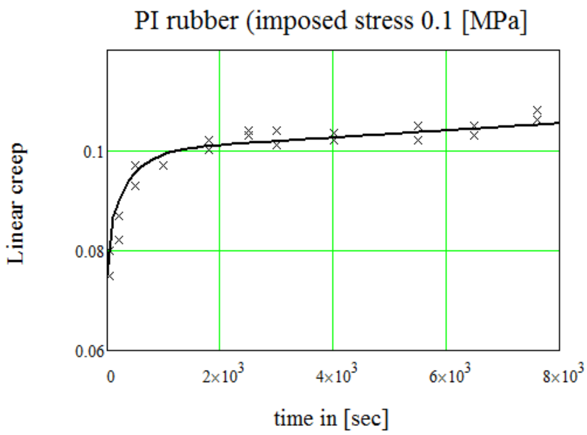

Figure 3. Linear creep curves

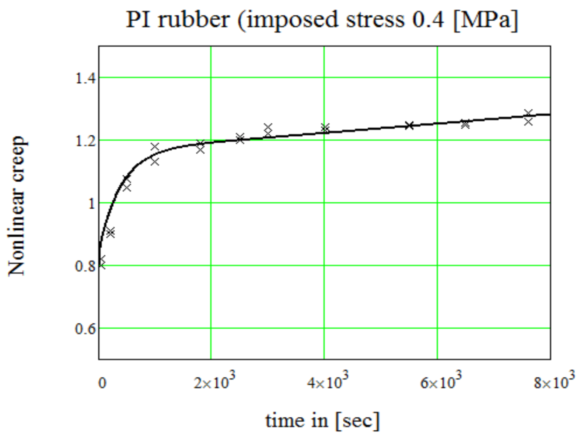

Figure 4. Nonlinear creep curves

In all the figures the experimental data are plotted with stars.

4. CONCLUSION

The proposed sum of relaxation kernels in the hereditary theory to predict the mechanical behavior of viscoelastic solids well describe the creep and stress relaxation curves in large time interval. The authors demonstrated that if the resolving kernel of one singular stress relaxation kernel in the respective integral equation is known, the resolving creep kernel of the proposed sum of stress relaxation kernels represent a sum of the respective creep resolving kernels. This approach works well due to the great number of parameters and the singularity of the kernels.

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