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PROMISING TECHNIQUE FOR ANALYTIC TREATMENT OF DIFFUSION EQUATIONS

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Abstract: In this paper, the new powerful and efficient technique named Reconstruction of Variational Iteration Method (RVIM) is applied to find the analytic and approximate solutions for nonlinear diffusion equations. The RVIM technique is independent of any small parameters at all. The algorithm overcomes the difficulty arising in calculating nonlinear intricately terms. Besides, it provides us with a simple way to ensure the convergence of solution series, so that we can always get enough accuracy in approximations as well as this method is capable of reducing the size of calculation. The obtained numerical results compared with the analytic solutions show that the method provides remarkable accuracy for different values of time (t) and distance (x).
Keywords: Reconstruction of Variational Iteration Method (RVIM), Analytic solution, Diffusion Equations

1. INTRODUCTION

Differential equations are widely used to describe physical problems. In most cases, the exact solution of these problems may not be available. In addition, it is much easier computing and analyzing these solutions by means of the numerical methods without wasting time or spending money for experimenting problems. Alternatively, the numerical methods can provide approximate solutions rather than the exact solutions. But most of these methods have low accuracy and are highly time consuming. Reaching to a high accurate approximation for linear and nonlinear equations has always been important while it challenges tasks in science and engineering. Therefore several numbers of approximate methods have been established like Homotopy perturbation Method (HPM) Variational Iteration Method (VIM) and many other methods so on each of which has advantages and disadvantages [1-12]. We introduce a new analytical method of nonlinear problems called the reconstruction of variational iteration method, which in the case of comparing with VIM [1-4] and HPM [5-8], not uses Lagrange multiplier as variational methods do and not requires small parameter in equations as the perturbation techniques. RVIM has been shown to solve a large class of nonlinear problems with approximations converging to solutions rapidly, effectively, easily, and accurately. The method used gives rapidly convergent successive approximations. As stated before, we aim to achieve analytic solutions to problems. We also aim to approve that the reconstruction of variational iteration method is powerful, efficient, and promising in handling scientific and engineering problems.

2. BASIC IDEA OF RVIM

For convenience of the reader, to clarify the basic idea of our proposed method in [15], we consider the following differential equation:

$$\mathrm{Lu}(x_1,\cdots,x_n)+\mathrm{Nu}(x_1,\cdots,x_n)=f(x_1,\cdots,x_n) \tag{1}$$

Suppose that

$$\mathrm{Lu}(x_1,\cdots,x_n)=\sum_{i=0}^{n} L_{x_i} u(x_i) \tag{2}$$

where L is a linear operator, N a nonlinear operator and $f(x_1,\cdots,x_n)$ an inhomogeneous term.
We can rewrite Eq. (1) as follows:

$$L_{x_j} u(x_1,\cdots,x_n)=\underbrace{f(x_1,\cdots,x_n)-\mathrm{Nu}(x_1,\cdots,x_n)-\sum_{\substack{i=0\\ i\neq j}}^{n} L_{x_i} u(x_i)}_{w(u(x_1,\cdots,x_n))} \tag{3}$$

Therefore L_(x_j) u(x_1,⋯,x_n)=w(u(x_1,⋯,x_n))

The Reconstruction of variational iteration method assumes a series solution for $u(x,t)$ given by an infinite sum of components

$$u(x_1,\cdots,x_n)=\lim_{n\to\infty} u_n(x_1,\cdots,x_n)=\lim_{n\to\infty}\sum_{i=0}^{n} v_i(x_1,\cdots,x_n)$$

Where v_0 is the solution of $L_{x_j}u=0$, with initial conditions of the main problem,

$$v_1(x_1,\cdots,x_n)=\varphi(v_0);\, v_{i+1}(x_1,\cdots,x_n)=\varphi\left(\sum_{i=0}^{i} v_i(x_1,\cdots,x_n)\right)-\sum_{i=1}^{i} v_i(x_1,\cdots,x_n)\,,\qquad i\geq 1$$

Which that $\varphi(v_i)$ is obtained as

$$L_{x_j}\varphi(v_i)=w(u(x_1,\cdots,x_n)) \tag{4}$$

Therefore by taking Laplace transform of both sides of the Eq. (4) in the usual way and using the artificial initial conditions equal to zero in case of finding $\varphi(v_i)$, we obtain the result as follows

$$P(s).\Phi_i\left(x_1,\cdots,x_{i-1},s,x_{i+1},x_n\right)=\varpi\left(v_i\left(x_1,\cdots,x_{i-1},s,x_{i+1},x_n\right)\right) \tag{5}$$

where $L[\varphi(v_i)]=\Phi_i$, $P(s)$ is a polynomial with the degree of the highest derivative in Eq. (5), The same as the highest order of the linear operator L_{x_j}. So that

$$L[w]=\varpi \tag{6-a}$$

$$\Psi(s)=\frac{1}{P(s)} \tag{6-b}$$

$$L[\psi(x_i)]=\Psi(s) \tag{6-c}$$

In Eq. (6-a) the function $\varpi\left(v_i\left(x_1,\cdots,x_{i-1},s,x_{i+1},x_n\right)\right)$ and $w\left(v_i\left(x_1,\cdots,x_{i-1},x_i,x_{i+1},x_n\right)\right)$ have been abbreviated as ϖ, w respectively. So, rewrite Eq. (5) as;

$$\Phi_i\left(x_1,\cdots,x_{i-1},s,x_{i+1},x_n\right)=\varpi\left(v_i\left(x_1,\cdots,x_{i-1},s,x_{i+1},x_n\right)\right).\Psi(s) \tag{7}$$

Now, by applying the inverse Laplace Transform to both sides of Eq. (7) and using the Convolution Theorem, we have;

$$\varphi(v_i)=\int_0^{x_i} w\left(v_i\left(x_1,\cdots,x_{i-1},\tau,x_{i+1},x_n\right)\right).\psi(x_i-\tau)d\tau \tag{8}$$

Therefore

$$u_{n+1}(x_1,\cdots,x_n)=\sum_{i=0}^{n+1} v_i=u_0(x_1,\cdots,x_n)+\int_0^{x_i} w\left(u_n\left(x_1,\cdots,x_{i-1},\tau,x_{i+1},x_n\right)\right).\psi(x_i-\tau)d\tau \tag{9}$$

Identifying the initial approximation of u_0, the remaining approximations u_n, $n>0$ can be determined such that each terms is determined by using the previous terms, and the approximation of iteration formula entirely will be evaluated. Consequently, the exact solution could be obtained by using:

$$u=\lim_{n\to\infty} u_n=\lim_{n\to\infty}\sum_{i=0}^{n} v_i \tag{10}$$

In what follows, we will apply the RVIM method to homogeneous/non-homogeneous, linear and nonlinear diffusion equations to illustrate the strength of this method and to establish exact solutions for these problems.

3. APPLICATION OF RVIM TO DIFFUSION EQUATIONS

In this section, we consider studying the nonlinear diffusion equations [16-19]

Example 1.

$$u_t=(u_x u^2)_x \tag{11}$$

$$u(x,0)=\frac{x+a}{2c} \tag{12}$$

where a and c are arbitrary constants. This equation models a slow diffusion process such as evaporation and melting [15]. Here, auxiliary linear operator is selected as $L_t u(x,t)=u_t$. By using the Eq. (11) we have the following operator form equation:

$$L_t u(x,t)=u_t=\overbrace{(u_x u^2)_x}^{w(u(x,t))} \tag{13}$$

Therefore $\varphi(v_i)$ is defined as

$$\varphi(v_i)=\int_0^t w\left(v_i(x,\tau)\right)d\tau \tag{14}$$

Then by using the Eq. (14), the **RVIM** method formula in t-direction for calculation of the approximate solution of equation (10), can be readily obtained as

$$u_{n+1}(x,t)=\sum_{i=0}^{i} v_i(x,t)=u_0(x,t)+\int_0^t[(u_n(x,\tau))^2.u_x(x,\tau)]_x\,d\tau \tag{15}$$

Whereas, the initial approximation must be satisfy with the following equations

$$L_t u(x,t)=0, \quad u(x,0)=\frac{x+a}{2c}$$

Therefore we begin with $u_0(x,t)=v_0=\frac{x+a}{2c}$, accordingly by the equation (15) one can get the higher order approximation of the exact solution as the following relations;

$$u_1(x,t)=\sum_{i=0}^{1} v_i(x,t)=\frac{x+a}{2c}+\frac{x+a}{4c^3}t$$

$$u_2(x,t)=\sum_{i=0}^{2} v_i(x,t)=\frac{x+a}{2c}+\frac{x+a}{4c^3}t+\frac{3}{16}\frac{x+a}{c^5}t^2+\frac{1}{16}\frac{x+a}{c^7}t^3+\frac{1}{128}\frac{x+a}{c^9}t^4$$

The remaining approximations $u_n(x,t), n>3$ can be completely determined such that each term is determined by using previous term, and thus, the n^{th} solution is given in the closed form.

$$u(x,t)=\lim_{n\to\infty} u_n(x,t)=\lim_{n\to\infty}\sum_{i=0}^{n} v_i(x,t)=\frac{x+a}{2\sqrt{c^2-t}}, \qquad t<c^2 \tag{16}$$

Table 1. The numerical results for $u_4(x,t)$ in comparison with the exact solution $u(x,t)$ when a=2, c=4

	x	Exact solution	Numerical solution	Absolute error
t=0.1	0.1	0.2633241778	0.2633241779	1.446735283E−34
	0.2	0.2758634244	0.2758634244	4.599377423E−32
	0.3	0.2884026710	0.2884026709	0.
	0.4	0.3009419176	0.3009419175	1.000000000E−10
	0.5	0.3134811641	0.3134811641	3.672200000E−11
t=0.3	0.1	0.2649960942	0.2649960942	0.
	0.2	0.2776149558	0.2776149558	0.
	0.3	0.2902338175	0.2902338173	1.000000000E−10
	0.4	0.3028526791	0.3028526791	4.075811508E−11
	0.5	0.3154715408	0.3154715407	1.134868300E−10
t=0.5	0.1	0.2667002667	0.2667002665	2.000000000E−10
	0.2	0.2794002794	0.2794002792	3.000000000E−10
	0.3	0.2921002921	0.2921002919	3.000000000E−10
	0.4	0.3048003048	0.3048003045	2.084616600E−10
	0.5	0.3175003175	0.3175003173	2.000000000E−10
t=1	0.1	0.2711088342	0.2711088281	6.16413416208 E-9
	0.2	0.2840187788	0.2840187723	6.45766436028 E-9
	0.3	0.2969287231	0.2969287164	6.75119455846 E-9
	0.4	0.3098386677	0.3098386607	7.04472475667 E-9
	0.5	0.3227486122	0.3227486049	7.33825495487E-9
t=1.5	0.1	0.2757435090	0.2757434583	5.072010000000 E-8
	0.2	0.2888741522	0.2888740992	5.313543813713 E-8
	0.3	0.3020047956	0.3020047402	5.555077271956 E-8
	0.4	0.3151354388	0.3151353809	5.796580135960 E-8
	0.5	0.3282660821	0.3282660218	6.038110000000 E-8
t=2.5	0.1	0.2806243040	0.2857730326	7.7068000000 E-7
	0.2	0.2939873660	0.2993812723	8.0736000000 E-7
	0.3	0.3073504282	0.3129895120	8.4408000000 E-7
	0.4	0.3207134903	0.3265977516	8.8075000000 E-7
	0.5	0.3340765525	0.3402059913	9.1747158760 E-7
t=3	0.1	0.2857738034	0.2912155114	0.000002091643
	0.2	0.2993820797	0.3050829167	0.000002191246
	0.3	0.3129903560	0.3189503221	0.000002290847
	0.4	0.3265986324	0.3328177272	0.000002390449
	0.5	0.3402069088	0.3466851325	0.000002490052

To verify numerically whether the proposed RVIM method leads to higher accuracy, we can evaluate the numerical solutions. Using n^{th} approximation shows the high degree of accuracy and u_n the n^{th} approximation is accurate for quite low of n $(n=4)$. From the obtained numerical result summarized in Table 1 we conclude that the method, RVIM method for slow diffusion equation, gives remarkable accuracy. The behavior of the solutions obtained by the RVIM method is shown for different values of time in comparison with exact solution, Table 2 and Table 3.

Table 2. The numerical results for $u_7(x,t)$ in comparison withthe exact solution $u(x,t)$ when h=1, c=1

x	Absolute error [t = 0.1]	Absolute error [t = 0.3]	Absolute error [t = 0.5]
1	1.000000000 E -9	5.400000000 E -8	0.0000106050000
2	1.000000000 E -9	8.000000000 E -8	0.0000159070000
3	1.272493000 E -9	1.069690000 E -7	0.0000212078400
4	2.000000000 E -9	1.340000000 E -7	0.0000265120000
5	2.000000000 E -9	1.620000000 E -7	0.0000318160000
6	1.000000000 E -9	1.880000000 E -7	0.0000371160000
7	2.000000000 E -9	2.140000000 E -7	0.0000424220000
8	2.000000000 E -9	2.410000000 E -7	0.0000477210000
9	2.750000000 E -9	2.692500000 E -7	0.0000530220000
10	4.000000000 E -9	2.960000000 E -7	0.0000583247453

Table 3. The numerical results for $u_7(x,t)$ in comparison with the exact solution $u(x,t)$ when c=10

x	Absolute error [t = 0.1]	Absolute error [t = 0.3]	Absolute error [t = 0.5]
1	9.984046216 E -13	7.884295969 E -10	8.252386953 E -8
2	1.039936184 E -10	2.953718377 E -9	3.301954779 E -7
3	2.089856415 E -10	7.095866433 E -9	7.427148233 E -7
4	1.015974474E -9	1.161487360 E -8	0.000001320381914
5	2.496011554 E -11	1.871073992 E -8	0.000002063096735
6	1.035942566 E -9	2.738346543 E -8	0.000002971859303
7	1.048921827 E -9	3.663305031 E -8	0.000004045669683
8	1.063897896 E -9	4.745949396 E -8	0.000005284527655
9	2.080870775 E -9	6.186279738 E -8	0.000006682433509
10	9.984046216 E -11	7.884295969 E -8	0.000008252386953

Example 2.

$$u_t=(u_x u^2)_x \tag{17}$$

With initial condition

$$u(x,0)=\frac{x+h}{2\sqrt{c}},$$

where $h, c>0$, are arbitrary contacts.
At first, auxiliary linear operator is selected as

$$L_t u(x,t)=u_t=\overbrace{(u_x u^2)_x}^{w(u(x,t))} \tag{18}$$

Therefore $\varphi(v_i)$ is defined as

$$\varphi(v_i)=\int_0^t w(v_i(x,\tau))d\tau \tag{19}$$

So using the Eq. (18) and (19) we obtain the following RVIM's iteration formula in t-direction:

$$u_{n+1}(x,t)=\sum_{i=0}^{n+1} v_i(x,t)=u_0(x,t)+\int_0^t (u_n^2(x,\tau)u_{nx}(x,\tau))_x\, d\tau \tag{20}$$

The subscript n indicates the n^{th} approximation of the solution; we can obtain the other components with selecting the initial approximation as:

$$u_0(x,t)=\frac{x+h}{2\sqrt{c}} \tag{21}$$

So with the iteration formula (20), we obtain the following successive approximations

$$u_1(x,t)=\sum_{i=0}^{1} v_i(x,t)=\frac{x+h}{2\sqrt{c}}\left(1+\frac{t}{2c}\right)$$

$$u_2(x,t)=\sum_{i=0}^{2} v_i(x,t)=\frac{1}{128}\frac{(x+h)}{c^{9/2}}t^4+\frac{1}{16}\frac{(x+h)}{c^{7/2}}t^3+\frac{3}{16}\frac{(x+h)}{c^{5/2}}t^2+\frac{1}{4}\frac{(x+h)}{c^{3/2}}t+\frac{1}{2}\frac{(x+h)}{c^{1/2}}$$

And so on. In the same manner, the rest of components of the iteration Eq. (20) can be obtained. Therefore, the solution of u(x, t) in closed form is

$$u(x,t)=\lim_{n\to\infty} u_n(x,t)=\lim_{n\to\infty}\sum_{i=0}^{n} v_i(x,t)=\frac{x+h}{2\sqrt{c-t}}$$

The obtained numerical result is summarized in Table 2.

Example 3.

$$u_t=(u_x u)_x \tag{22}$$

$$u(x,0)=\frac{x^2}{c}$$

Where $a, c\neq 0$, are arbitrary contacts. At first, auxiliary linear operator is selected as

$$L_t u(x,t)=u_{tt}=\overbrace{(u_x u)_x}^{w(u(x,t))} \tag{23}$$

Therefore $\varphi(v_i)$ is defined as

$$\varphi(v_i)=\int_0^t w(v_i(x,\tau))d\tau \tag{24}$$

So RVIM's iteration formula in t-direction can be readily obtained.

$$u_{n+1}(x,t)=\sum_{i=0}^{n+1} v_i(x,t)=u_0(x,t)+\int_0^t (u_n(x,\tau)u_{nx}(x,\tau))_x\, d\tau \tag{25}$$

The subscript n indicates the n^{th} approximation; by considering the given initial values, we can select $u_0(x,t)=v_0=\frac{x^2}{c}$. By substituting $u_0(x,t)$ to Eq. (25), we have;

$$u_1(x,t)=\sum_{i=0}^{1} v_i(x,t)=x^2\left(\frac{1}{c}+\frac{6t}{c^2}\right)$$

$$u_2(x,t)=\sum_{i=0}^{2} v_i(x,t)=x^2\left(\frac{1}{c}+\frac{6t}{c^2}+\frac{36t^2}{c^3}+\frac{72t^3}{c^4}\right)$$

And so on. In the same manner, the rest of components of the iteration formula, Eq. (25) can be obtained. Therefore, the solution of u(x, t) in closed form is

$$u(x,t)=\lim_{n\to\infty} u_n(x,t)=\lim_{n\to\infty}\sum_{i=0}^{n} v_i(x,t)=\frac{x^2}{c-6t}$$

The obtained numerical result is summarized in Table 3.

Example 4.

$$u_t=(u_x u^{-1})_x \tag{26}$$

With initial condition,

$$u(x,0)=\frac{2c}{(x+a)^2}$$

where $a, c \neq 0$, are arbitrary contacts. As said before, auxiliary linear operator which that plays an important role in choosing the initial approximation of the solution, is selected as

$$L_t u(x,t)=u_{tt}=\left(u_x u^{-1}\right)_x=w(u(x,t) \tag{27}$$

Using Eqs. (19) and (24), RVIM's iteration formula in t-direction can be readily obtained.

$$u_{n+1}(x,t)=\sum_{i=0}^{n+1} v_i(x,t)=u_0(x,t)+\int_0^t \left(u^{-1}(x,\tau)u_x(x,\tau)\right)_x d\tau \tag{28}$$

By this assumption that u_0 is the solution of ,$L_t u=0$, we start with the initial approximation as

$$u_0(x,t)=\frac{2c}{(x+a)^2} \tag{29}$$

And with the iteration Eq. (28), we obtain the following successive approximations

$$u_1(x,t)=\sum_{i=0}^{1} v_i(x,t)=\frac{2c}{(x+a)^2}+\frac{2t}{(x+a)^2}$$

$$u_2(x,t)=\sum_{i=0}^{2} v_i(x,t)=\frac{2c}{(x+a)^2}+\frac{2t}{(x+a)^2}$$

Continuing in this manner, we can obtain $u_{n+1}(x,t)=u_n(x,t)$ for $n>1$, which means that the exact solution of Eq. (21) is easily obtained in the form

$$u(x,t)=\lim_{n\to\infty} u_n(x,t)=\lim_{n\to\infty}\sum_{i=0}^{n} v_i(x,t)=\frac{2c}{(x+a)^2}+\frac{2t}{(x+a)^2}$$

4. CONCLUSION

In this paper, the RVIM method has been successfully applied to find the solution of the nonlinear diffusion equations governing the diffusion project and to show the power of this method and its significant features. It gives rapidly convergent successive approximations through using the RVIM's iteration relation without any restrictive assumptions or transformation that may change the physical behavior of the problems.

Moreover, RVIM reduces the size of calculations by not requiring the tedious Adomian polynomials, and hence the iteration is direct and straightforward. The solutions obtained by the RVIM method for appropriate initial conditions, can be, in turn, expressed in a closed form, the exact solution. The results reported here provide further evidence of the usefulness of RVIM for finding the analytic and numeric solutions for the linear and nonlinear diffusion equations and, it is also a promising method to solve different types of nonlinear equations in mathematical physics.

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