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THERMOELASTIC STRESSES IN NONHOMOGENEOUS PRISMATIC BARS

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Abstract: This paper deals with the determination of thermal stresses in nonhomogeneous prismatic bars by the application of theorem of minimum of complementary energy. The material properties and the applied thermal field do not depend on the axial coordinate. The presented analysis is valid for compound bars and bars made from functionally graded materials. The applied mechanical loads are bending moment and axial force at the end cross sections of the nonhomogeneous bar. An example illustrates the application of the presented formula to determine the normal stress field.
Keywords: thermoelastic, nonhomogeneous, complementary energy, prismatic bar

1. INTRODUCTION

This paper deals with the determination of thermal stresses in nonhomogeneous prismatic bars. The derivation of the formula for stresses caused by mechanical and thermal loads is based on the principle of minimum of complementary energy. The cross section of the bar is an arbitrary bounded plain domain and the material properties and the temperature field do not depend on the axial coordinate. The considered inhomogeneity means that the material properties are arbitrary functions of the cross-sectional coordinates. The presented analysis is valid for compound bars whose material properties are discontinous functions of the cross-sectional coordinates and bars made from functionally graded materials, whose material properties are smooth functions of the cross-sectional coordinates. If there are no prescribed surface displacements than the theorem of minimum of complementary energy can be formulated as in [1-3].

Among all the sets of admissible stresses σ_x, σ_y, σ_z, τ_{xy}, τ_{xy}, τ_{xy} which satisfy all the equilibrium and the prescribed stress boundary conditions, the set of actual stress components makes the functional $\tilde{\Pi}_c\left(\sigma_x,\sigma_y,\sigma_z,\tau_{xy},\tau_{xz},\tau_{zy}\right)$ defined by

$$\tilde{\Pi}_c=\int_V\left\{\frac{1}{2E}\left[\sigma_x^2+\sigma_y^2+\sigma_z^2-2\nu\left(\sigma_x\sigma_y+\sigma_y\sigma_z+\sigma_z\sigma_x\right)+2(1+\nu)(\tau_{xy}^2+\tau_{yz}^2+\tau_{zx}^2)\right]+\alpha T(\sigma_x+\sigma_y+\sigma_z)\right\}\mathrm{d}V \tag{1}$$

an absolute minimum.

In Eq. (1) σ_x, σ_y, σ_z are normal stresses, τ_{xy}, τ_{xy}, τ_{xy} are shearing stresses, E is the Young modulus, ν is the Poisson ratio, α is the coefficient of thermal expansion, $T=\theta-\theta_0$, where θ is the absolute temperature and θ_0 is the reference temperature at which the stresses are zero if the body is undeformed, V is the space domain occupied by the thermoelastic body (Figure 1).

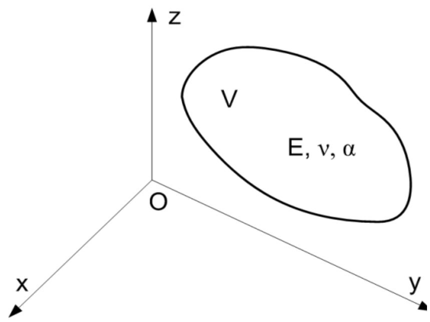

Figure 1. Thermoelastic body

The considered nonhomogeneous prismatic bar and its mechanical loads are shown in Figure 2, where $\mathbf{F}=F\mathbf{e}_z$ is the applied axial force and $\mathbf{M}=M_x\mathbf{e}_x+M_y\mathbf{e}_y$ is the applied bending moment. The material properties are functions of x and y, that is, we have $E=E(x,y)$ and $\alpha=\alpha(x,y)$. In our formulation in the considered thermoelastic bar problem the Poisson's ratio does not appear. The temperature difference field T also depends only on x and y, it is a given function. In the framework of strenght of materials the equilibrium stress field is characterized by the equations

$$\sigma_x=\sigma_y=\tau_{yz}=\tau_{xy}=\tau_{xz}=0, \qquad \sigma_z=\sigma_z(x,y), \tag{2}$$

$$K_1\left[\sigma_z(x,y)\right]=\int_A\sigma_z(x,y)\mathrm{d}A-F=0, \tag{3}$$

$$\mathbf{K}_2\left[\sigma_z(x,y)\right]=\int_A\mathbf{R}\sigma_z(x,y)\mathrm{d}A-\mathbf{e}_z\times\mathbf{M}=\mathbf{0}. \tag{4}$$

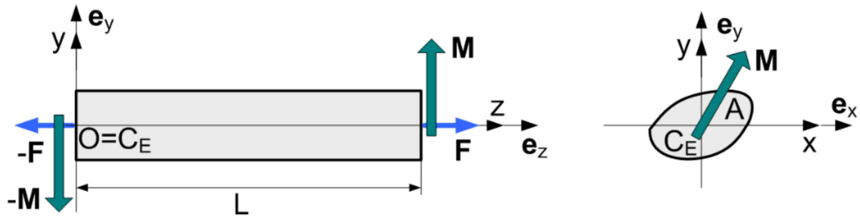

Figure 2. Nonhomogeneous prismatic bar

Eqs. (2-4) refer to the coordinate system *Oxyz* with unit vectors $\mathbf{e}_x$, $\mathbf{e}_y$, $\mathbf{e}_z$ and

$$\mathbf{R} = x\mathbf{e}_x + y\mathbf{e}_y, \tag{5}$$

the cross between two vectors in Eq. (4) denotes their vectorial product and the cross section of the nonhomogeneous bar is *A*. Here we note that axis *z* is the *E*-weighted center line of the nonhomogeneous bar, it connects of *E*-weighted centres of cross sections. The *E*-weighted centre C_E is defined by the next equation:

$$\int_A E(x,y)\mathbf{R}\,\mathrm{d}A = \mathbf{0}. \tag{6}$$

The state of stresses are independent of the axial coordinate *z* from this it follows that the axial force and bending moment do not change along axis *z*. The complementary energy of nonhomogeneous bar according to Eq. (1) is as follows

$$\tilde{\Pi}_c = L\int_A \left\{ \frac{1}{2E(x,y)}\left[\sigma_z(x,y)\right]^2 + \alpha(x,y)T(x,y)\sigma_z(x,y) \right\} \mathrm{d}A, \tag{7}$$

where *L* is the lenght of the bar (Figure 2). Let $\Pi_c\left[\sigma_z(x,y)\right]$ be defined as

$$\Pi_c = \int_A \left\{ \frac{1}{2E(x,y)}\left[\sigma_z(x,y)\right]^2 + \alpha(x,y)T(x,y)\sigma_z(x,y) \right\} \mathrm{d}A\,. \tag{8}$$

2. THE DETERMINATION OF STRESS FIELD

According to the minimum of complementary energy we look for the minimum of $\Pi_c\left[\sigma_z(x,y)\right]$ under the subsidiary conditions given by Eqs. (3), (4). The method of Lagrange multipliers will be used [4, 5]. We define a new functional which contains the constraints given by Eqs. (3) and (4)

$$F\left[\sigma_z(x,y),\, \lambda_1,\, \boldsymbol{\lambda}_2\right] = \Pi_c\left[\sigma_z(x,y)\right] - \lambda_1 K_1\left[\sigma_z(x,y)\right] - \boldsymbol{\lambda}_2\cdot\mathbf{K}_2\left[\sigma_z(x,y)\right]. \tag{9}$$

In Eq. (9) the scalar product of two vectors is indicated by dot. The necessary condition of minimum is formulated by the next varational equation

$$\begin{aligned}\delta F = &\int_A \left\{ \frac{\sigma_z(x,y)}{E(x,y)} + \alpha(x,y)T(x,y) - \lambda_1 - \boldsymbol{\lambda}_2\cdot\mathbf{R} \right\}\delta\sigma_z \mathrm{d}A - \delta\lambda_1\left[\int_A \sigma_z(x,y)\mathrm{d}A - F\right] \\ &-\delta\boldsymbol{\lambda}_2\cdot\left[\int_A \mathbf{R}\sigma_z(x,y)\mathrm{d}A - \mathbf{e}_z\times\mathbf{M}\right] = 0.\end{aligned} \tag{10}$$

Since $\delta\sigma_z$, $\delta\lambda_1$, $\delta\underline{\lambda}_2$ are arbitrary we obtain the following equations from Eq. (10)

$$\sigma_z(x,y) = E(x,y)\left[\lambda_1 + \boldsymbol{\lambda}_2\cdot\mathbf{R} - \alpha(x,y)T(x,y)\right], \tag{11}$$

$$F = \int_A \sigma_z(x,y)\mathrm{d}A, \quad \mathbf{e}_z\times\mathbf{M} = \int_A \mathbf{R}\sigma_z(x,y)\mathrm{d}A. \tag{12}$$

Combination of Eq. (6) with Eqs. (11) and (12) gives

$$\lambda_1 = \frac{F+N_T}{A_E}, \quad N_T = \int_A E(x,y)\alpha(x,y)T(x,y)\mathrm{d}A. \tag{13}$$

In Eq. $(13)_1$

$$A_E = \int_A E(x,y)\mathrm{d}A\,. \tag{14}$$

Substitution of Eq. $(12)_2$ into Eq. (11) yields the next expression

$$\lambda_1\int_A E(x,y)\mathbf{R}\mathrm{d}A + \boldsymbol{\lambda}_2\cdot\int_A E(x,y)\mathbf{R}\circ\mathbf{R}\mathrm{d}A - \int_A E(x,y)\alpha(x,y)\mathbf{R}T(x,y)\mathrm{d}A = \mathbf{e}_z\times\mathbf{M}. \tag{15}$$

Here the circle between two vectors denotes their tensorial (dyadic) product. We introduce the Euler tensor

$$\mathbf{I} = \int_A E(x,y)\mathbf{R}\circ\mathbf{R}\mathrm{d}A = I_y\mathbf{e}_x\circ\mathbf{e}_x + I_{xy}\left(\mathbf{e}_x\circ\mathbf{e}_y + \mathbf{e}_y\circ\mathbf{e}_x\right) + I_x\mathbf{e}_y\circ\mathbf{e}_y, \tag{16}$$

where

$$I_x = \int_A E(x,y)x^2 \mathrm{d}A, \quad I_{xy} = \int_A E(x,y)xy\mathrm{d}A, \quad I_y = \int_A E(x,y)y^2 \mathrm{d}A. \tag{17}$$

Let M_T be defined as

$$\mathbf{e}_z \times \mathbf{M}_T = \int_A E(x,y)\alpha(x,y)\mathbf{R}T(x,y)dA. \tag{18}$$

In Eq. (15) the coefficient of λ_1 vanishes, that is we have

$$\mathbf{I}\cdot\boldsymbol{\lambda}_2 = \mathbf{e}_z \times \mathbf{M}_T + \mathbf{e}_z \times \mathbf{M}. \tag{19}$$

Denote the unit vector in direction of $\boldsymbol{\lambda}_2$ is $\mathbf{m} = m_x\mathbf{e}_x + m_y\mathbf{e}_y$, which means that $\boldsymbol{\lambda}_2 = \lambda_2\mathbf{m}$. Let $\mathbf{n} = \mathbf{m}\times\mathbf{e}_z = n_x\mathbf{e}_x + n_y\mathbf{e}_y$ be. From Eq. (19) we get

$$\lambda_2(\mathbf{I}\cdot\mathbf{m})\times\mathbf{e}_z = \lambda_2\mathbf{I}\cdot\mathbf{n} = (\mathbf{e}_z\times\mathbf{M}_T)\times\mathbf{e}_z + (\mathbf{e}_z\times\mathbf{M})\times\mathbf{e}_z = \mathbf{M}_T + \mathbf{M}, \quad \lambda_2\mathbf{I}\cdot\mathbf{n} = \mathbf{M}_T + \mathbf{M}. \tag{20}$$

From Eq. (20) it follows that

$$\lambda_2\mathbf{n} = \mathbf{I}^{-1}\cdot(\mathbf{M}_T + \mathbf{M}). \tag{21}$$

Eq. (21) gives a possibility to obtain the unit vectos **n**

$$\mathbf{n} = \frac{\mathbf{I}^{-1}\cdot(\mathbf{M}_T + \mathbf{M})}{\left|\mathbf{I}^{-1}\cdot(\mathbf{M}_T + \mathbf{M})\right|}. \tag{22}$$

On the other hand from Eq. (20) we have

$$\lambda_2\mathbf{n}\cdot\mathbf{I}\cdot\mathbf{n} = \mathbf{M}_T\cdot\mathbf{n} + \mathbf{M}\cdot\mathbf{n}, \tag{23}$$

that is

$$\lambda_2 = \frac{M_{Tn} + M_n}{I_n}, \quad I_n = \mathbf{n}\cdot\mathbf{I}\cdot\mathbf{n} = I_x n_y^2 + 2I_{xy}n_y n_x + I_y n_x^2 \quad M_{Tn} = \mathbf{M}_T\cdot\mathbf{n}, \; M_n = \mathbf{M}\cdot\mathbf{n}. \tag{24}$$

In Eq. (11)

$$\boldsymbol{\lambda}_2\cdot\mathbf{R} = \lambda_2\mathbf{m}\cdot\mathbf{R} = \lambda_2(\mathbf{e}_z\times\mathbf{n})\cdot\mathbf{R} = \lambda_2(xm_x + ym_y) = \lambda_2(yn_x - xn_y). \tag{25}$$

Summarizing the results obtained the following formula can be derived for the σ_z

$$\sigma_z(x,y) = E(x,y)\left\{\frac{F + N_T}{A_E} + \frac{M_n + M_{Tn}}{I_n}(yn_x - xn_y) - \alpha T\right\}. \tag{26}$$

3. EXAMPLE

The cross section of the considered bar is shown in Figure 3. This cross section is made of two different homogeneous materials with Young moduli $E_1 = E$, $E_2 = 3E$ and the coefficients of linear thermal expansion $\alpha_1 = \alpha$, $\alpha_2 = 2\alpha$. There are no applied mechanical loads, that is $\mathbf{F} = \mathbf{0}$, $\mathbf{M} = \mathbf{0}$. The temperature difference T=constant on the whole cross section. For homogeneous cross section the uniform temperature does not produce any stress field, the homogeneous bar is stress free. The position of the E-weighted centre of cross section is given in Figure 3. The elements of Euler tensor are

$$I_x = 514Ec^4, \; I_y = 64Ec^4, \; I_{xy} = -96Ec^4. \tag{27}$$

A simple computation gives

$$A_E = 48Ec^2. \tag{28}$$

From Eq. (18) we get

$$\mathbf{M}_T = T\left[\int_A E(x,y)\alpha(x,y)\mathbf{R}\mathrm{d}A\right]\times\mathbf{e}_z = 24E\alpha Tc^3\mathbf{e}_x + 48E\alpha Tc^3\mathbf{e}_y. \tag{29}$$

Application of formula (22) gives

$$\mathbf{n} = 0.9751\mathbf{e}_x + 0.2216\mathbf{e}_y. \tag{30}$$

From Eq. (16) and Eq. $(24)_2$ it follows that

$$I_n = I_x n_y^2 + 2I_{xy}n_y n_x + I_y n_x^2 = 44.5073Ec^4. \tag{31}$$

Eq. $(13)_2$ for N_T gives

$$N_T = 72E\alpha Tc^2, \tag{32}$$

and from Eqs. $(24)_4$ and (29) we get

$$M_{Tn} = 34.036E\alpha Tc^3. \tag{33}$$

By the use of above computed values we can determine the stress field of composite bar caused by uniform temperature field. We determine the normal stresses at points $P(-4c, 2c)$ and $Q(8c, -2c)$. The computation gives

$$\sigma_z(R) = 20.002 E\alpha T, \tag{34}$$

$$\sigma_z(Q) = -5.0413 E\alpha T. \tag{35}$$

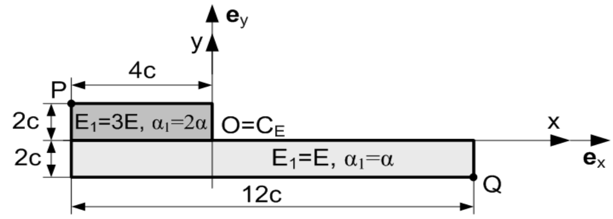

Figure 3. Nonhomogeneous cross section

4. SUPPLEMENTARY NOTE

In this section the mechanical meaning of the Lagrange multipliers λ_1 and $\boldsymbol{\lambda}_2$ will be given. The stress-strain relation for one-dimensional problem of thermoelasticity is formulated as [6, 7]

$$\sigma_z(x,y) = E(x,y)\left[\varepsilon_z(x,y) - \alpha T(x,y)\right]. \tag{36}$$

In Eq. (36) ε_z is the normal strain. Comparison of Eq. (11) with Eq. (36) gives

$$\varepsilon_z(x,y) = \lambda_1 + \boldsymbol{\lambda}_2 \cdot \mathbf{R} = \lambda_1 + \lambda_2 \mathbf{m} \cdot \mathbf{R}. \tag{37}$$

Eq. (37) shows that λ_1 is normal strain at the E-weighted centre C_E of cross section and $\boldsymbol{\lambda}_2$ is the curvature vector of the deformed E-weighted centre line, that is

$$\varepsilon_z(0,0) = \lambda_1 = \varepsilon_0, \qquad \boldsymbol{\lambda}_2 = \kappa \mathbf{m}, \tag{38}$$

where κ is the curvature of the deformed longitudinal fiber determined by x=0, y=0 and $0 \le z \le L$. The thermoelastic pure bending problem of nonhomogeneous prismatic bars, based on the Euler-Bernoulli beam theory was analysed by Stokes [8]. Stokes paper uses a direct approach starting from the assumed form of normal strain which is

$$\varepsilon_z(x,y) = \frac{\eta - \eta_n}{R}. \tag{39}$$

Here

$$\eta = \mathbf{m} \cdot \mathbf{R}, \qquad \kappa = \frac{1}{R}, \qquad \eta_N = -R\varepsilon_0. \tag{40}$$

The zero line of longitudinal strains is given by η_N. Our approach is different from the one presented by Stokes [8]. It demonstrates the efficiency of the variation method in solving the problems of mechanics of solids.

5. CONCLUSIONS

The theorem of minimum of complementary energy is used to get the formula of thermal stresses in nonhomogeneous prismatic bars. The solution of formulated variational problem is based on the application of Lagrange multipliers. A presented example illustrates that the nonhomogeneous prismatic bar under the action of uniform temperature field is not stress free.

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