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BUCKLING ANALYSIS OF ORTHOTROPIC DOUBLE- NANOPLATE- SYSTEMS BASED ON NONLOCAL TWO- VARIABLE REFINED PLATE THEORY

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Abstract: This paper presents an analytical study on the buckling of orthotropic double- nanoplate-system (DNPS) subjected to biaxial compression using the two-variable refined plate theory and nonlocal elasticity theory. The two - variable refined plate theory takes into account the effects of small scale and transverse shear strains through the thickness of the nanoplate. The theory assumes parabolic distribution of the transverse shear strains through thickness of the nanoplate and does not require shear correction factors. Derived nonlocal governing equations are based on the nonlocal theory and the principle of virtual displacements. The expression for buckling load of a simply supported orthotropic DNPS was derived using the Navier's method.
Keywords: Double-orthotropic nanoplates, Refined plate theory, Nonlocal buckling; Nonlocal elasticity theory; Navier's method

1. INTRODUCTION

The discovery of carbon nanotube by Lijima [1] represents a significant breakthrough in the development of nanoscience and nanotechnology. It has been experimentally proven that three dimensional graphite can be shelled into micron-sized graphene sheets with the thickness of several atomic layers or even a single atomic layer [2]. Due to their superb electronic, chemical properties and high mechanical strength, graphene sheets (GSs) are often used as component of micro electro-mechanical systems (MEMS) and nano electro-mechanical systems (NEMS). After the introduction of these novel methods forgraphene sheets preparation [3] numerous investigations have been conducted on electronic and mechanical properties of single layered graphene sheets (SLGSs). There are numerous studies which deal with the investigation of electronic properties of graphene sheets while on the other side, only a small number of investigations focuses on their mechanical properties [4]. Superior mechanical, electrical and chemical properties of GSs have allowed their application in a number of novel structures and devices at nanoscale [5]. Exact prediction of vibrational and buckling behaviour of nanostructural elements (nanorods, nanobeams, carbon nanotubes, and nanoplates, such as GSs) is crucial to their engineering and manufacturing. However, experimental investigations on nanoscale size samples are very difficult and expensive. Application of molecular-dynamic (MD) simulation is also difficult and financially exacting, especially when analyzing nanostructures comprising a larger number of atoms. This motivates the development of an appropriate mathematical model which would allow analysis of mechanical behaviour of nanostructures, to allow their successful application in MEMS and NEMS. Atomistic methods have been used in analysis of nanostructure behaviour [6] and yielded good results.

Classical continuum elasticity is a scale-free theory and cannot predict the influence of size effects on mechanical behaviour of nanostructures. However, some researchers have used classical plate theory to investigate vibrations in graphenenanosheets [7] and buckling carbon nanotubes [8]. It should be noted that, in some cases, application of classical continuum mechanics leads to controversial and inexact results. This requires some modifications of the classical continuum theory so that it can be used for analysis of small scale structures. Eringen suggested a nonlocal continuum elasticity [9] which considers the size effect in a way which allows the study of size-dependent mechanical properties. This theory assumes that the stress at a considered point does not depend solely on the deformation at that particular point, but is also the function of deformations at all points of the entire domain of the considered body.

Peddiessonet. al [10] have been the first to apply the theory of nonlocal elasticity on the analysis of static deformations of Euler-Bernoulli nanobeams. Recently, a significant number of contributions emerged that use the nonlocal constitutive Eringen equations to analyze flexural [11], vibrational [12], and buckling [13] behaviours of nanoplates. Moreover, there are contributions [14] which analyze shape optimization of nanorods and nanobeams.This paper presents a formulation of nonlocal two-variable refined plate theory for the buckling analysis of biaxially compressed orthotropic DNPS. Stress displacement relations of the orthotropic DNPS are derived on the basis of the nonlocal constitutive relations of Eringen [9] and the basic assumptions of a two-variable refined plate theory. A recent contribution [15] reports on a study of biaxial buckling of isotropic DNPS based on nonlocal classical Kirchhof's plate theory. To the best of the author's knowledge, up to present, there are no available studies on biaxial buckling phenomenon in orthotropic DNPS which are based on the two-variable refined plate theory.

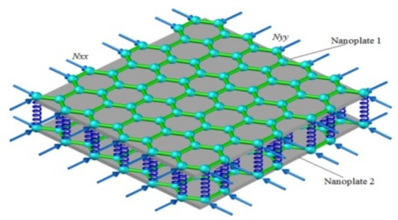

Figure1. Double-orthotropic nanoplates coupled by an elastic medium

2. MATHEMATICAL FORMULATION

Let us consider nanoplates the length of a , width b and height h, as shown in Figure 1.

2.1. Nonlocal constitutive relations

Within the nonlocal theory of elasticity, stress value at reference point within elastic continuum does not solely depend on the value of deformation at that particular point, but also depends on deformations at all other points of the elastic body. Based on [9], a differential nonlocal constitutive equation can be written in which the internal length characteristic can be simply considered as material parameter

$$\left(1-g^2\nabla^2\right)\sigma_{ij}^{nl}=\sigma_{ij}^{l},\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2},g=e_0 l \quad (1)$$

where g represents nonlocal small scale parameter which depends on material constant, e_0, and internal length characteristic, l. σ_{ij}^{nl} and σ_{ij}^{l} are components of nonlocal and local stress tensor, respectively.

2.2. Governing equations for double-orthotropic nanoplate

We are considering a typical double-orthotropic nanoplate shown in Fig.1. The two nanoplates are referred to as nanoplate 1 and nanoplate 2. The two nanoplates of orthotropic DNPS are coupled by an elastic medium (polymer matrix). The elastic medium between two nanoplates was mathematically modelled using a system of vertical Winkler springs. The springs are oriented orthogonally relative to nanoplates and have stiffness, k. In this study, various stiffness values have been used for different elastic media (polymer matrices). It is assumed that the two nanoplates have the same length, width and bending rigidity. The displacements of the nanoplate 1 and nanoplate 2 are denoted by $w_{b1}(x,y,t)$, $w_{s1}(x,y,t)$ and $w_{b2}(x,y,t)$, $w_{s2}(x,y,t)$ respectively.

Nanoplate-1

$$\begin{aligned}&D_{11}\frac{\partial^4 w_{b1}}{\partial x^4}+2\left(D_{12}+2D_{66}\right)\frac{\partial^4 w_{b1}}{\partial x^2\partial y^2}+D_{22}\frac{\partial^4 w_{b1}}{\partial y^4}+N_0\left(\frac{\partial^2 w_{b1}}{\partial x^2}+\frac{\partial^2 w_{s1}}{\partial x^2}\right)+RN_0\left(\frac{\partial^2 w_{b1}}{\partial y^2}+\frac{\partial^2 w_{s1}}{\partial y^2}\right)+\\&k(w_{b1}+w_{s1}-w_{b2}-w_{s2})-N_0g^2\nabla^2\left(\frac{\partial^2 w_{b1}}{\partial x^2}+\frac{\partial^2 w_{s1}}{\partial x^2}\right)-RN_0g^2\nabla^2\left(\frac{\partial^2 w_{b1}}{\partial y^2}+\frac{\partial^2 w_{s1}}{\partial y^2}\right)-\\&kg^2\nabla^2(w_{b1}+w_{s1}-w_{b2}-w_{s2})=0\end{aligned}\quad (2)$$

$$\begin{aligned}&\frac{1}{84}\left(D_{11}\frac{\partial^4 w_{s1}}{\partial x^4}+2\left(D_{12}+2D_{66}\right)\frac{\partial^4 w_{s1}}{\partial x^2\partial y^2}+D_{22}\frac{\partial^4 w_{s1}}{\partial y^4}\right)-\left[A_{55}\frac{\partial^2 w_{s1}}{\partial x^2}+A_{44}\frac{\partial^2 w_{s1}}{\partial y^2}\right]+\\&N_0\left(\frac{\partial^2 w_{b1}}{\partial x^2}+\frac{\partial^2 w_{s1}}{\partial x^2}\right)+RN_0\left(\frac{\partial^2 w_{b1}}{\partial y^2}+\frac{\partial^2 w_{s1}}{\partial y^2}\right)+k(w_{b1}+w_{s1}-w_{b2}-w_{s2})-\\&N_0g^2\nabla^2\left(\frac{\partial^2 w_{b1}}{\partial x^2}+\frac{\partial^2 w_{s1}}{\partial x^2}\right)-RN_0g^2\nabla^2\left(\frac{\partial^2 w_{b1}}{\partial y^2}+\frac{\partial^2 w_{s1}}{\partial y^2}\right)-kg^2\nabla^2(w_{b1}+w_{s1}-w_{b2}-w_{s2})=0\end{aligned}\quad (3)$$

Nanoplate-2

$$\begin{aligned}&D_{11}\frac{\partial^4 w_{b2}}{\partial x^4}+2\left(D_{12}+2D_{66}\right)\frac{\partial^4 w_{b2}}{\partial x^2\partial y^2}+D_{22}\frac{\partial^4 w_{b2}}{\partial y^4}+N_0\left(\frac{\partial^2 w_{b2}}{\partial x^2}+\frac{\partial^2 w_{s2}}{\partial x^2}\right)+RN_0\left(\frac{\partial^2 w_{b2}}{\partial y^2}+\frac{\partial^2 w_{s2}}{\partial y^2}\right)+\\&k(w_{b2}+w_{s2}-w_{b1}-w_{s1})-N_0g^2\nabla^2\left(\frac{\partial^2 w_{b2}}{\partial x^2}+\frac{\partial^2 w_{s2}}{\partial x^2}\right)-RN_0g^2\nabla^2\left(\frac{\partial^2 w_{b2}}{\partial y^2}+\frac{\partial^2 w_{s2}}{\partial y^2}\right)-\\&kg^2\nabla^2(w_{b2}+w_{s2}-w_{b1}-w_{s1})=0\end{aligned}\quad (4)$$

$$\begin{aligned}&\frac{1}{84}\left(D_{11}\frac{\partial^4 w_{s2}}{\partial x^4}+2\left(D_{12}+2D_{66}\right)\frac{\partial^4 w_{s2}}{\partial x^2\partial y^2}+D_{22}\frac{\partial^4 w_{s2}}{\partial y^4}\right)-\left[A_{55}\frac{\partial^2 w_{s2}}{\partial x^2}+A_{44}\frac{\partial^2 w_{s2}}{\partial y^2}\right]+\\&N_0\left(\frac{\partial^2 w_{b2}}{\partial x^2}+\frac{\partial^2 w_{s2}}{\partial x^2}\right)+RN_0\left(\frac{\partial^2 w_{b2}}{\partial y^2}+\frac{\partial^2 w_{s2}}{\partial y^2}\right)+k(w_{b2}+w_{s2}-w_{b1}-w_{s1})-\\&N_0g^2\nabla^2\left(\frac{\partial^2 w_{b2}}{\partial x^2}+\frac{\partial^2 w_{s2}}{\partial x^2}\right)-RN_0g^2\nabla^2\left(\frac{\partial^2 w_{b2}}{\partial y^2}+\frac{\partial^2 w_{s2}}{\partial y^2}\right)-\\&kg^2\nabla^2(w_{b2}+w_{s2}-w_{b1}-w_{s1})=0\end{aligned}\quad (5)$$

For the solution of Eqs. (2),(3),(4) and (5) we change variables by considering w_b and w_s as the displacement of the nanoplate-1 relative to nanoplate-2

$$w_b = w_{b1} - w_{b2} \tag{6}$$

$$w_s = w_{s1} - w_{s2} \tag{7}$$

Subtracting Eq. (4) from (2) and Eq. (5) from (3) would lead to

$$\begin{aligned} &D_{11}\frac{\partial^4 w_b}{\partial x^4} + 2\left(D_{12}+2D_{66}\right)\frac{\partial^4 w_b}{\partial x^2 \partial y^2} + D_{22}\frac{\partial^4 w_b}{\partial y^4} + N_0\left(\frac{\partial^2 w_b}{\partial x^2}+\frac{\partial^2 w_s}{\partial x^2}\right) + RN_0\left(\frac{\partial^2 w_b}{\partial y^2}+\frac{\partial^2 w_s}{\partial y^2}\right)+ \\ &2k(w_b+w_s) - N_0 g^2\nabla^2\left(\frac{\partial^2 w_b}{\partial x^2}+\frac{\partial^2 w_s}{\partial x^2}\right) - RN_0 g^2\nabla^2\left(\frac{\partial^2 w_b}{\partial y^2}+\frac{\partial^2 w_s}{\partial y^2}\right)- \\ &2kg^2\nabla^2(w_b+w_s)=0 \end{aligned} \tag{8}$$

$$\begin{aligned} &\frac{1}{84}\left(D_{11}\frac{\partial^4 w_s}{\partial x^4} + 2\left(D_{12}+2D_{66}\right)\frac{\partial^4 w_s}{\partial x^2 \partial y^2} + D_{22}\frac{\partial^4 w_s}{\partial y^4}\right) - \left[A_{55}\frac{\partial^2 w_s}{\partial x^2} + A_{44}\frac{\partial^2 w_s}{\partial y^2}\right] + N_0\left(\frac{\partial^2 w_b}{\partial x^2}+\frac{\partial^2 w_s}{\partial x^2}\right)+ \\ &RN_0\left(\frac{\partial^2 w_b}{\partial y^2}+\frac{\partial^2 w_s}{\partial y^2}\right) + 2k(w_b+w_s) - N_0 g^2\nabla^2\left(\frac{\partial^2 w_b}{\partial x^2}+\frac{\partial^2 w_s}{\partial x^2}\right)- \\ &RN_0 g^2\nabla^2\left(\frac{\partial^2 w_b}{\partial y^2}+\frac{\partial^2 w_s}{\partial y^2}\right) \end{aligned} \tag{9}$$

Utilizing Eqs. (6) and (7) in Eqs. (4) and (5) we get

$$\begin{aligned} &D_{11}\frac{\partial^4 w_{b2}}{\partial x^4} + 2\left(D_{12}+2D_{66}\right)\frac{\partial^4 w_{b2}}{\partial x^2 \partial y^2} + D_{22}\frac{\partial^4 w_{b2}}{\partial y^4} + N_0\left(\frac{\partial^2 w_{b2}}{\partial x^2}+\frac{\partial^2 w_{s2}}{\partial x^2}\right) + RN_0\left(\frac{\partial^2 w_{b2}}{\partial y^2}+\frac{\partial^2 w_{s2}}{\partial y^2}\right)- \\ &k(w_b+w_s) - N_0 g^2\nabla^2\left(\frac{\partial^2 w_{b2}}{\partial x^2}+\frac{\partial^2 w_{s2}}{\partial x^2}\right) - RN_0 g^2\nabla^2\left(\frac{\partial^2 w_{b2}}{\partial y^2}+\frac{\partial^2 w_{s2}}{\partial y^2}\right)+ \\ &kg^2\nabla^2(w_b+w_s)=0 \end{aligned} \tag{10}$$

$$\begin{aligned} &\frac{1}{84}\left(D_{11}\frac{\partial^4 w_{s2}}{\partial x^4} + 2\left(D_{12}+2D_{66}\right)\frac{\partial^4 w_{s2}}{\partial x^2 \partial y^2} + D_{22}\frac{\partial^4 w_{s2}}{\partial y^4}\right) - \left[A_{55}\frac{\partial^2 w_{s2}}{\partial x^2} + A_{44}\frac{\partial^2 w_{s2}}{\partial y^2}\right]+ \\ &N_0\left(\frac{\partial^2 w_{b2}}{\partial x^2}+\frac{\partial^2 w_{s2}}{\partial x^2}\right) + RN_0\left(\frac{\partial^2 w_{b2}}{\partial y^2}+\frac{\partial^2 w_{s2}}{\partial y^2}\right) - k(w_b+w_s)- \\ &N_0 g^2\nabla^2\left(\frac{\partial^2 w_{b2}}{\partial x^2}+\frac{\partial^2 w_{s2}}{\partial x^2}\right) - RN_0 g^2\nabla^2\left(\frac{\partial^2 w_{b2}}{\partial y^2}+\frac{\partial^2 w_{s2}}{\partial y^2}\right)+ \\ &kg^2\nabla^2(w_b+w_s)=0 \end{aligned} \tag{11}$$

The critical buckling load of a simply supported, orthotropic, rectangular double-nanoplate-system, will be determined in this paper by using Navier's solution. Navier's method is only used for simply supported boundary conditions on all four edges of the rectangular nanoplate. At each end of the nanoplates, the displacements and the nonlocal moments are considered to be zero.

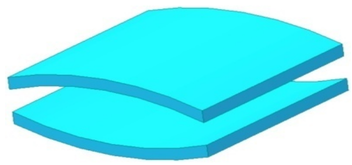

Figure 2. Asynchronous type buckling

3. EXACT SOLUTION OD THE BUCKLING EQUATIONS

To discuss the buckling of double-orthotropic nanoplates, three typical cases such as asynchronous (out-of-phase) sequence of buckling, synchronous (in phase) sequence of buckling and one stationary nanoplate are considered.

3.1. Out-of- phase buckling

In this case, both nanoplates are buckled out-of-phase (asynchronous):

$$\begin{aligned} &w_{b1}(x,y,t) - w_{b2}(x,y,t) \neq 0, \\ &w_{s1}(x,y,t) - w_{s2}(x,y,t) \neq 0 \end{aligned} \tag{12}$$

The configuration of the orthotropic DNPS with the out-of phase sequence of buckling is shown in Fig.2. To analyze out-of-phase buckling, Eqs. (8) and (9) are used. These equations with nonlocal boundary conditions can be solved by the Navier' method assuming solutions in the form of

$$w_b = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} w_{bmn}\sin(\alpha x)\sin(\beta y) \tag{13}$$

$$w_s = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} w_{smn}\sin(\alpha x)\sin(\beta y) \tag{14}$$

In the upper equation:

$$\alpha = \frac{m\pi}{a}, \qquad \beta = \frac{n\pi}{b} \tag{15}$$

where m and n are the half waves numbers.

Substituting Eqs. (12) and (13) into Eqs. (8) and (9), the following system of equations is obtained

$$\begin{bmatrix} B_1 + B_2 N_0 & B_3 + B_4 N_0 \\ B_3 + B_4 N_0 & B_5 + B_6 N_0 \end{bmatrix} \begin{Bmatrix} w_{bmn} \\ w_{smn} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{16}$$

$$B_1 = D_{11}\alpha^4 + 2(D_{12} + 2D_{66})\alpha^2\beta^2 + D_{22}\beta^4 + 2k + 2kg^2(\alpha^2 + \beta^2) \tag{17a}$$

$$B_2 = -\left[(\alpha^2 + R\beta^2) + g^2(\alpha^4 + (1+R)\alpha^2\beta^2 + R\beta^4)\right] \tag{17b}$$

$$B_3 = 2k + 2kg^2(\alpha^2 + \beta^2) \tag{18a}$$

$$B_4 = -\left[(\alpha^2 + R\beta^2) + g^2(\alpha^4 + (1+R)\alpha^2\beta^2 + R\beta^4)\right] \tag{18b}$$

$$B_5 = \frac{1}{84}\left[D_{11}\alpha^4 + 2(D_{12} + 2D_{66})\alpha^2\beta^2 + D_{22}\beta^4\right] + \left(A_{55}\alpha^2 + A_{44}\beta^2\right) + 2k + 2kg^2(\alpha^2 + \beta^2) \tag{19a}$$

$$B_6 = -\left[(\alpha^2 + R\beta^2) + g^2(\alpha^4 + (1+R)\alpha^2\beta^2 + R\beta^4)\right] \tag{19b}$$

It should be noted that

$$B_2 = B_4 = B_6$$

For nontrivial solution of w_{bmn} and w_{smn} , the determinant of the coefficient matrix in Eq. (16) must be zero

$$\begin{vmatrix} B_1 + B_2 N_0 & B_3 + B_4 N_0 \\ B_3 + B_4 N_0 & B_5 + B_6 N_0 \end{vmatrix} = 0 \tag{20}$$

Solving this determinant yields

$$N_{0Asyn} = \frac{B_3^2 - B_1 B_5}{B_2(B_1 + B_5 - 2B_3)} \tag{21}$$

The non-dimensional form of critical buckling load can be written as

$$\tilde{N}_{Asyn} = N_0 \frac{a^2}{D_{11}} \tag{22}$$

3.2. In-phase buckling

In the case of in-phase buckling, two nanoplates are buckled synchronously, thus the relative displacement between them disappears, i.e., ($w_{b1}(x,y,t) - w_{b2}(x,y,t) = 0$, $w_{s1}(x,y,t) - w_{s2}(x,y,t) = 0$). The configuration of the orthotropic DNPS with the in-phase sequence of buckling is shown in Fig.3. To analyze out-of-phase buckling Eqs. (10) and (11) are used.

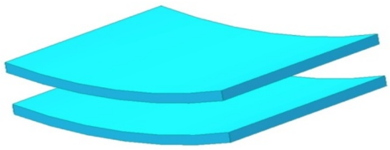

Figure 3. Synchronous type buckling

Substituting Eqs. (13) and (14) into Eqs. (10) and (11) the following system of equations is obtained

$$\begin{bmatrix} C_1 + C_2 N_0 & C_3 N_0 \\ C_3 N_0 & C_4 + C_5 N_0 \end{bmatrix} \begin{Bmatrix} w_{bmn} \\ w_{smn} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{23}$$

where

$$C_1 = D_{11}\alpha^4 + 2(D_{12} + 2D_{66})\alpha^2\beta^2 + D_{22}\beta^4 \tag{24a}$$

$$C_2 = -\left[(\alpha^2 + R\beta^2) + g^2(\alpha^4 + (1+R)\alpha^2\beta^2 + R\beta^4)\right] \tag{24b}$$

$$C_3 = -\left[(\alpha^2 + R\beta^2) + g^2(\alpha^4 + (1+R)\alpha^2\beta^2 + R\beta^4)\right] \tag{24c}$$

$$C_4 = \frac{1}{84}\left[D_{11}\alpha^4 + 2(D_{12} + 2D_{66})\alpha^2\beta^2 + D_{22}\beta^4\right] + \left(A_{55}\alpha^2 + A_{44}\beta^2\right) \tag{25a}$$

$$C_5 = -\left[(\alpha^2 + R\beta^2) + g^2(\alpha^4 + (1+R)\alpha^2\beta^2 + R\beta^4)\right] \tag{25b}$$

It should be noted that

$$C_2 = C_3 = C_5$$

Following procedure similar to that of out-of-phase buckling, one can determine critical buckling load as

$$N_{0Syn} = \frac{-C_1 C_4}{C_2\left(C_1 + C_4\right)} \tag{26}$$

Here, we see that the critical buckling load is independent of the stiffness of the coupling springs and therefore the orthotropic DNPS can be treated as a single orthotropic nanoplate. The non-dimensional form of critical buckling load can be written as

$$\tilde{N}_{syn} = N_0 \frac{a^2}{D_{11}} \tag{27}$$

3.3. One nanoplate being stationary

Another buckling mode of interest is one nanoplate being stationary, i.e., ($w_{b2}(x,y,t) = 0, w_{s2}(x,y,t) = 0$). The configuration of the orthotropic DNPS with one plate being stationary is shown in Fig.4. In this case, Eqs. (2) and (3) are reduced to

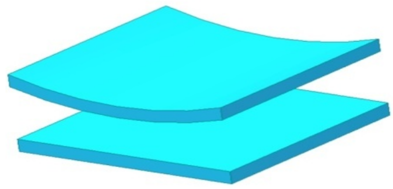

Figure 4. Buckling with one nanoplate fixed

$$\begin{aligned} &D_{11}\frac{\partial^4 w_{b1}}{\partial x^4} + 2\left(D_{12} + 2D_{66}\right)\frac{\partial^4 w_{b1}}{\partial x^2 \partial y^2} + D_{22}\frac{\partial^4 w_{b1}}{\partial y^4} + N_0\left(\frac{\partial^2 w_{b1}}{\partial x^2} + \frac{\partial^2 w_{s1}}{\partial x^2}\right) + RN_0\left(\frac{\partial^2 w_{b1}}{\partial y^2} + \frac{\partial^2 w_{s1}}{\partial y^2}\right) + \\ &k(w_{b1} + w_{s1}) - N_0 g^2 \nabla^2\left(\frac{\partial^2 w_{b1}}{\partial x^2} + \frac{\partial^2 w_{s1}}{\partial x^2}\right) - RN_0 g^2 \nabla^2\left(\frac{\partial^2 w_{b1}}{\partial y^2} + \frac{\partial^2 w_{s1}}{\partial y^2}\right) - \\ &kg^2 \nabla^2 (w_{b1} + w_{s1}) = 0 \end{aligned} \tag{28}$$

$$\begin{aligned} &\frac{1}{84}\left(D_{11}\frac{\partial^4 w_{s1}}{\partial x^4} + 2\left(D_{12} + 2D_{66}\right)\frac{\partial^4 w_{s1}}{\partial x^2 \partial y^2} + D_{22}\frac{\partial^4 w_{s1}}{\partial y^4}\right) - \left[A_{55}\frac{\partial^2 w_{s1}}{\partial x^2} + A_{44}\frac{\partial^2 w_{s1}}{\partial y^2}\right] + \\ &N_0\left(\frac{\partial^2 w_{b1}}{\partial x^2} + \frac{\partial^2 w_{s1}}{\partial x^2}\right) + RN_0\left(\frac{\partial^2 w_{b1}}{\partial y^2} + \frac{\partial^2 w_{s1}}{\partial y^2}\right) + k(w_{b1} + w_{s1}) - \\ &N_0 g^2 \nabla^2\left(\frac{\partial^2 w_{b1}}{\partial x^2} + \frac{\partial^2 w_{s1}}{\partial x^2}\right) - RN_0 g^2 \nabla^2\left(\frac{\partial^2 w_{b1}}{\partial y^2} + \frac{\partial^2 w_{s1}}{\partial y^2}\right) - \\ &kg^2 \nabla^2 (w_{b1} + w_{s1}) = 0 \end{aligned} \tag{29}$$

Here DNPS behaves as a nanoplate embedded or supported on an elastic medium. If we suppose solution, Eqs. (28) and (29) have the same form as Eqs. (13) and (14) the following system of equations is obtained

$$\begin{bmatrix} E_1 + E_2 N_0 & E_3 + E_4 N_0 \\ E_3 + E_4 N_0 & E_5 + E_6 N_0 \end{bmatrix}\begin{Bmatrix} w_{bmn} \\ w_{smn} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{30}$$

where

$$E_1 = D_{11}\alpha^4 + 2(D_{12} + 2D_{66})\alpha^2\beta^2 + D_{22}\beta^4 + k + kg^2(\alpha^2 + \beta^2) \tag{31a}$$

$$E_2 = -\left[(\alpha^2 + R\beta^2) + g^2(\alpha^4 + (1+R)\alpha^2\beta^2 + R\beta^4)\right] \tag{31b}$$

$$E_3 = k + kg^2(\alpha^2 + \beta^2) \tag{32a}$$

$$E_4 = -\left[(\alpha^2 + R\beta^2) + g^2(\alpha^4 + (1+R)\alpha^2\beta^2 + R\beta^4)\right] \tag{32b}$$

$$\begin{aligned} E_5 = &\frac{1}{84}\left[D_{11}\alpha^4 + 2(D_{12} + 2D_{66})\alpha^2\beta^2 + D_{22}\beta^4\right] + \left(A_{55}\alpha^2 + A_{44}\beta^2\right) + \\ &k + kg^2(\alpha^2 + \beta^2) \end{aligned} \tag{33a}$$

$$E_6 = -\left[(\alpha^2 + R\beta^2) + g^2(\alpha^4 + (1+R)\alpha^2\beta^2 + R\beta^4)\right] \tag{33b}$$

It should be noted that

$$E_2 = E_4 = E_6$$

Following procedure similar to that of out-of-phase buckling, one can determine critical buckling load as

$$N_{0St} = \frac{E_3^2 - E_1 E_5}{E_2(E_1 + E_5 - 2E_3)} \tag{34}$$

The non-dimensional form of critical buckling load can now be written as

$$\tilde{N}_{St} = N_0 \frac{a^2}{D_{11}} \tag{35}$$

4. NUMERICAL RESULTS AND DISCUSSION

For illustration, the properties of single-layered orthotropic graphene sheets are considered. The two single-layered orthotropic graphene sheets are coupled by embedded polymer. The Young's modulus and Poisson's ratio of the orthotropic graphene sheets are $E_1 = 1765\,GPa, E_2 = 1588\,GPa,$ and $\nu_{12} = 0.3, \nu_{21} = 0.27,$. The thickness of the graphene sheets is $h = 0.34\,nm$ .The elastic medium which binds two nanoplates was modelled as a system of springs whose stiffness, k, is varied within the 10-100 range . The range of nonlocal small scale parameter for graphene sheet is adopted as g=0-2nm.

The buckling analysis of the single isotropic nanoplate using two-variable refined plate theory. It is well known that in the Case 2 (in-phase buckling) the two bonded nanoplates act like one. The same holds true if for the Case 1 (out-of-phase buckling) and Case 3 (one nanoplate fixed) we assume stiffness parameter k=0. The properties of the nanoplates in the validation analysis were adopted for the case of isotropic nanoplates, as in [15]. Young's modulus of the nanoplate was adopted as $E_1=E_2=1.06$ TPa, Poisson's ratio $\nu_{12}=\nu_{21}=0.25$ and thickness h=0.34 nm. In [13] analysis of buckling of a biaxially compressed orthotropic single-layered graphene sheet was conducted based on the classical Kirchhoff's plate theory and analytical expression for nondimensional buckling load was given. It has been shown that in Case 3 (one-nanoplate fixed), orthotropic DNPS behaves like a single-layered nanoplate. Thus, in this paper the accuracy of the presented theory was checked, as shown in Fig. 5, by comparing the analytical solution for Case 3 (one-nanoplate fixed) with the solution obtained in [13]. Application of the classical Kirchhoff's plate theory yields expressions for the two remaining cases (out-of-phase buckling and in-phase buckling) for orthotropic DNPS. Although the expressions were derived, they are here omitted for brevity. Shown in Fig. 5 is the good matching of results for all three buckling cases.

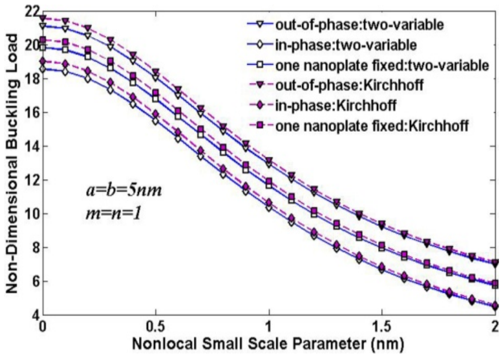

Figure 5. Effect nonlocal small scale parameter on non-dimensional buckling load for two-variable refined plate theory and Kirchhoff's plate theory for m=n=1

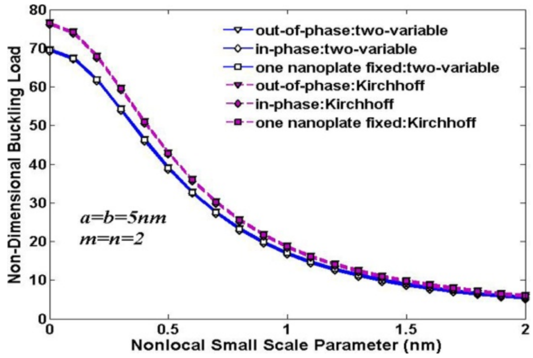

Figure 6. Effect nonlocal small scaleparameter on non-dimensional buckling load for two-variable refined and Kirchhoff's plate theory for m=n=2

Shown in Fig. 5 is comparison between nondimensional buckling load obtained by nonlocal small scale parameter for two-variable refined plate theory and the classical Kirchhoff's plate theory. Square nanoplates of edges a=b=5 nm are considered. The stiffness parameter of the coupling springs between the nanoplates is assumed to be constant (k=25). The number of half waves is m=1, n=1. It can be seen that nondimensional buckling load in the case of two-variable refined plate theory is smaller than that obtained by the classical Kirchhoff's plate theory. The conclusion is that transverse shear deformation has the effect of reducing the buckling load. The difference in nondimensional buckling load gets smaller with the increase of nonlocal small scale parameter. It should be noted that larger values of nonlocal small scale parameter reduce the nondimensional buckling load, which is confirmed by previous contributions [15,13]. Comparing the three typical cases of buckling, it is possible to note that the value of nondimensional buckling load in Case 1 (out-of-phase buckling) is larger than that obtained in Case 3 (one-nanoplate fixed) and Case 2 (in-phase buckling). This corresponds to the results previously reported for isotropic DNPS [16]. Fig. 6 illustrates the difference in values of nondimensional buckling load in case when m=n=2. It can be seen that larger m and n result in larger nondimensional buckling load. Also, larger m and n lead to more pronounced difference in nondimensional buckling loads between results obtained by two-variable refined plate theory and classical Kirchhoff's plate theory. That difference is particularly noticeable for small values of nonlocal small scale parameter. In case when nonlocal small scale parameter is larger than one, the results for nondimensional buckling load yielded by the two theories become very close. Furthermore, Fig. 6 shows that in case when m=n=2, the values of nondimensional buckling load for the three typical cases of buckling are almost identical, which complies with [15] which considers isotropic DNPS based on the classical Kirchhoff's plate theory.

5. NUMERICAL RESULTS AND DISCUSSION

In the presented paper, analysis of buckling behaviour of the double-orthotropic-nanoplate-systems was performed on the basis of the nonlocal theory of elasticity and two-variable refined plate theory. Two single-layered orthotropic graphene sheets coupled by polymer matrix were considered, while the buckling load was obtained analytically for three typical deformation modes. Based on the numerical results, the following main conclusions are drawn. It was found that the values of nondimensional buckling load - obtained by two-variable refined plate theory - were smaller than the values obtained by the classical Kirchoff's plate theory. The difference in nondimensional buckling obtained by the two theories increases with the larger number of modes. The influence of nonlocal effect on the nondimensional buckling load is diminished by higher stiffness of coupling springs.

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