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THE NUMERICAL SOLUTION OF MHD BLASIUS FLOW ALONG WITH DIFFERENTIAL TRANSFORMATION METHOD AND PADE APPROXIMANT

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Abstract: The Magnetohydrodynamics (MHD) Blasius boundary layer flow over a flat plate in the presence of transverse magnetic field is studied in this paper. The approximate solution and skin friction coefficient of MHD boundary layer flow are obtained by using Runge-Kutta fourth order along with shooting technique and a method that couples the differential transform method (DTM) with Padé approximation called DTM-Padé. The approximate solutions are presented with the help of graphs and a table. The results of previous authors as particular cases are obtained confirming our results. It is suggested that inclusion of forcing force in the boundary layer equation needs higher dimension Padé approximants for better accuracy.
Keywords: Blasius equation, Shooting technique, Padé Approximants, Differential Transformation Method (DTM), Boundary Layers

1. INTRODUCTION

Nonlinear differential equations are usually arising from mathematical modeling of many physical systems. Some of them are solved using numerical methods and some are solved using analytical methods such as perturbation ([1],[2]). The numerical methods such as Runge-Kutta method are based on discritization techniques and they only permit us to calculate the approximate solutions for some values of time and space variables, which cause us to overlook some important phenomena, in addition to the intensive computer time required to solve problem. Thus, it is often costly and time consuming to get a complete curve of results and so in these methods, stability and convergence should be considered so as to avoid divergence or inappropriate results. Numerical difficulties additionally appear if a nonlinear problem contain singularities or has multiple solutions. Perturbation techniques are based on the existence of small / large parameters, the so- called perturbation quantity. Unfortunately, many nonlinear problems in science and engineering do not contain such kind of perturbation quantities at all. Some non-perturbative techniques, such as the artificial small parameter method [3], the Adomain's decomposition method [4] etc. have been developed. Different from perturbation techniques, these non-perturbative methods are independent upon small parameters. However, both the methods themselves cannot provide us with a simple way to adjust and control the convergence region and rate of given approximate series.

The flow and heat transfer of a viscous and incompressible fluid induced by a continuously moving or stretching surface in a resting fluid is relevant to many manufacturing processes such as polymers involves the cooling of continuous strips or filaments by drawing them through a quiescent fluid [5]. Further, glass blowing, casting of metals and spinning of fibers involves the flow due to a stretching surface. Crane [6] was first to study the boundary-layer flow due to stretching surface in an ambient fluid and applied a similarity transformation for steady boundary-layer flow by stretching of a sheet when its velocity varying linearly with the distance from a fixed point. He [7] considered the influence of heat transfer in flow over a stretching surface in the case where the temperature difference between the surface and the ambient fluid is proportional to a power of distance from a fixed point [8].

Magnetohydrodynamics (MHD) is the study of the interaction of conducting fluids with electromagnetic phenomena. The flow of an electrically conducting fluid in the presence of a magnetic field is important in various areas of technology and engineering such as MHD power generation, MHD flow meters, and MHD pumps (Hayat et al. ([9], [10]), Abdelkhalek [11]). As the MHD flow finds a lot of applications in industries and scientific research, the main goal of the present study is to extend the work of Peker et al. [12] which was confined to the flow without magnetic interaction. Moreover, in the result and discussion we have presented some recent works as particular cases in conformity with the present method of solution.

2.BLASIUS MAGNETIC FLOW

Consider a steady two dimensional MHD boundary layer flow of a viscous incompressible electrically conducting fluid past a thin flat plate which is placed in the direction of a uniform velocity U_∞. Let the origin of the co-ordinate be at leading edge of the plate, the $x-$axis be the direction of the uniform stream and the $y-$axis normal to the plate. An uniform transverse magnetic field of strength B_0 has been applied perpendicular to the plate. The Prandtl boundary layer equations subject to above considerations are

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{1}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\upsilon\frac{\partial^2 u}{\partial y^2}-\frac{\sigma B_0{}^2 u}{\rho}, \tag{2}$$

$$\left.\begin{array}{l} at\ \ y=0:\ \ u=v=0, \\ at\ \ y\to\infty: u\to U_\infty, \\ at\ x=0:\ \ u\to U_\infty \end{array}\right\} \tag{3}$$

where u and v are the velocity components parallel and perpendicular to the plate respectively.
To study the boundary layer flow the following similarity transformations are used;

$$\eta=y\sqrt{\frac{U_\infty}{\upsilon x}},\ \psi=\sqrt{\upsilon x U_\infty}\,f(\eta),$$

where $\eta(x,y)$ is the similarity variable and $\psi(x,y)$ is the stream function.
The velocity components u and v are defined by

$$(u,v)=\left(\frac{\partial\psi}{\partial y},-\frac{\partial\psi}{\partial x}\right)$$

The momentum equation (2) and the boundary condition (3) can be written as

$$f'''(\eta)+\frac{1}{2}f(\eta)f''(\eta)-Mf'(\eta)=0, \tag{4}$$

$$f(0)=0,\ f'(0)=0, f'(\infty)=1, 0<\eta<\infty. \tag{5}$$

where $M=\frac{\sigma B_0{}^2}{\rho}$, the magnetic parameter, σ and B_0 are respectively, the electrical conductivity and the magnetic induction.

However, Equations (4), (5) and other forms of Blasius problem have boundary conditions in unbounded domains and this creates difficulty for application of numerical methods for solution. To overcome this difficulty, Pade approximants and the differential transformation can be applied for numerical solution.

3. DIFFERENTIAL TRANSFORMATION METHOD

Differential transformation method is a numerical method based on Taylor expansion. This method tries to find the coefficients of series expansion of unknown function by using the initial data on the problem. The concept of differential transformation method was first proposed by Zhou [13]. It was applied to electric circuit analysis problems. Afterwards, it was applied to several systems and differential equations. For instance, initial value problems [14], difference equations [15], integro-diifferential equations [16], and partial differential equations [17], system of ordinary differential equations [18].

Definition 1.

The one dimensional differential transform of a function $y(x)$ at the point $x=x_0$ is defined as follows:

$$Y(k)=\frac{1}{k!}\left[\frac{d^k}{dx^k}y(x)\right]_{x=x_0}, \tag{6}$$

where $y(x)$ is the original function and $Y(k)$ is the transformed function.

Definition 2.

The differential inverse transform of $Y(k)$ is defined as follows:

$$y(x)=\sum_{k=0}^{\infty}Y(k)(x-x_0)^k. \tag{7}$$

$$y(x)=\sum_{k=0}^{\infty}\frac{1}{k!}\left[\frac{d^k}{dx^k}y(x)\right]_{x=x_0}(x-x_0)^k. \tag{8}$$

The following theorems are deduced from definitions as follows:

Theorem 1. If $f(x)=g(x)\pm h(x)$, then $F(k)=G(k)\pm H(k)$.

Theorem 2. If $f(x)=\lambda g(x)$, then $F(k)=\lambda G(k)$.

Theorem 3. If $f(x) = g(x)h(x)$, then $F(k) = G(k) \otimes H(k) = \sum_{k_1=0}^{k} G(k_1)H(k-k_1)$.

Theorem 4. If $f(x) = \dfrac{d^n g(x)}{dx^n}$, then $F(k) = \dfrac{(k+n)!}{k!} G(k+n)$.

Theorem 5. If $f(x) = g(x)\dfrac{d^2 h(x)}{dx^2}$, then $F(k) = G(k) \otimes P(k) = \sum_{r=0}^{k} (k-r+1)(k-r+2)F(r)P(k-r+2)$.

4. PADE APPROXIMANT

Some techniques exist to accelerate the convergence of a given series. Among them the so-called Pade approximant is widely applied (Baker and Morris, [19]). Suppose that a function $f(\eta)$ is represented by a power series,

$$f(\eta) = \sum_{i=0}^{\infty} c_i \eta^i \tag{9}$$

This expression is the fundamental starting point of any analysis using Pade approximants. The notation $c_i, i = 0,1,2----$ is reserved for the given set of coefficients and $f(\eta)$ is the associated function. $[L/M]$ Pade approximant is a rational fraction,

$$\frac{a_0 + a_1\eta + a_2\eta^2 ----a_L\eta^L}{b_0 + b_1\eta + b_2\eta^2 ----b_M\eta^M}, \tag{10}$$

which has a Maclaurin expansion, agrees with equation (9) as far as possible. It is noticed that in equation (10) there are L+1 numerator and M+1 denominator coefficients. So there are L+1 independent numerator and M independent denominator coefficients, making L+M+1 unknown coefficients in all. This number suggests that normally $[L/M]$ ought to fit the power series equation (9) through the orders $1, \eta, \eta^2 ------\eta^{L+M}$. In the notation of formal power series

$$\sum_{i=0}^{\infty} c_i \eta^i = \frac{a_0 + a_1\eta + a_2\eta^2 ----a_L\eta^L}{b_0 + b_1\eta + b_2\eta^2 ----b_M\eta^M} + o(\eta^{L+M+1}) \tag{11}$$

$$\begin{aligned}&\left(b_0 + b_1\eta + ---+ b_M\eta^M\right)\left(c_0 + c_1\eta + --\right)\\ &\qquad = a_0 + a_1\eta + -- b_L\eta^L + o(\eta^{L+M+1})\end{aligned} \tag{12}$$

Equating the coefficients of $\eta^{L+1}, \eta^{L+2} ---, \eta^{L+m}$ we get,

$$\left.\begin{array}{l} b_M c_{L-M+1} + b_{M-1}c_{L-M+2} + --- + b_0 c_{L+1} = 0, \\ b_M c_{L-M+2} + b_{M-1}c_{L-M+3} + --- + b_0 c_{L+2} = 0, \\ -------------------- \\ -------------------- \\ b_M c_L \quad + b_{M-1}c_{L+1} \quad + --- + b_0 c_{L+M} = 0, \end{array}\right\} \tag{13}$$

If $j < 0$, we define $c_i = 0$ for consistency. Since $b_0 = 1$, equation (13) become a set of M linear equations for M unknown denominator coefficients.

$$\begin{pmatrix} c_{L-M+1} & c_{L-M+2} & -- & c_{L+1} \\ c_{L-M+2} & c_{L-M+3} & -- & c_{L+2} \\ -- & -- & -- & -- \\ c_L & c_{L+1} & -- & c_{L+M} \end{pmatrix} \begin{pmatrix} b_M \\ b_{M-1} \\ -- \\ b_1 \end{pmatrix} = - \begin{pmatrix} c_{L+1} \\ c_{L+2} \\ -- \\ c_{L+M} \end{pmatrix} \tag{14}$$

From these equations, b_i may be found. The numerator coefficients $a_0, a_1, ----, a_L$, follow immediately from equation (12) by equating the coefficients of $1, \eta, \eta^2, ---, \eta^{L+M}$ such as,

$$\left.\begin{array}{l} a_0 = c_0, \\ a_1 = c_1 + b_1 c_0, \\ a_2 = c_2 + b_1 c_1 + b_2 c_0, \\ -----------, \\ a_L = c_L + \sum_{i=1}^{\min[L/M]} b_i c_{L-i}. \end{array}\right\} \tag{15}$$

Thus equations (14) and (15) normally determine the Pade numerator and denominator and are called Pade equations. The [L/M] Pade approximant is constructed which agrees with the equation (11) through order η^{L+M}.

5. SOLUTION OF THE PROBLEM

Consider the equation (4) with boundary conditions of a classical Blasius flow (Wazwaz [20]),

$$f'''(\eta)+\frac{1}{2}f(\eta)f''(\eta)-Mf'(\eta)=0, \tag{16}$$

$$f(0)=0,\ f'(0)=1, f'(-\infty)=0, -\infty<\eta<0. \tag{17}$$

Combination of the series obtained by DTM and Pade approximant will yield the numerical value of $f''(0)$ so as to reduce the present boundary value problem (BVP) into initial value problem (IVP). The diagonal Pade approximants of degree $[2/2]$ is used to determine the approximate solution of generalized MHD Blasius equation for different values of magnetic parameter. In particular, $M=0.0$ gives rise to the solution of Blasius equation.

Let $f''(0)=A$, where A is a positive constant. Now, the differential transform method (DTM) will be applied to equation (16) as follows.

$$(k+1)(k+2)(k+3)F(k+3)\quad +\frac{1}{2}\sum_{r=0}^{k}(k-r+1)(k-r+2)F(r)F(k-r+2)-M(k+1)F(k+1)=0 \tag{18}$$

The differential transform of boundary conditions are as follows,

$$F(0)=0, F(1)=1, F(2)=A/2. \tag{19}$$

Applying the differential inverse transform we get,

$$f(\eta)=\sum_{k=0}^{\infty}F(k)\eta^k \ =\eta+\tfrac{A}{2}\eta^2+\tfrac{M}{6}\eta^3+\tfrac{A(2M-1)}{48}\eta^4+\tfrac{(2M^2-2M-A^2)}{240}\eta^5+\tfrac{1}{120}(\tfrac{M^2A}{6}-\tfrac{2MA}{3}+\tfrac{A}{8})\eta^6$$
$$+\tfrac{1}{210}(\tfrac{M^3}{24}-\tfrac{5M^2}{24}-\tfrac{MA^2}{6}+\tfrac{M}{12}+\tfrac{11A^2}{96})\eta^7+\tfrac{1}{336}(\tfrac{M^3}{120}-\tfrac{39M^2A}{240}+\tfrac{5MA}{48}+\tfrac{A^3}{480}-\tfrac{A}{64})\eta^8+---- \tag{20}$$

Case I: ($M=0$)

The DTM expression (20) gives,

$$f(\eta)=\eta+\tfrac{A}{2}\eta^2-\tfrac{A}{48}\eta^4-\tfrac{A^2}{240}\eta^5+\tfrac{A}{960}\eta^6+\tfrac{11}{20160}\eta^7+(-\tfrac{A}{21504}+\tfrac{A^3}{161280})\eta^8$$
$$-\tfrac{43}{967680}\eta^9+(\tfrac{A}{552960}-\tfrac{5A^3}{387072})\eta^{10}+----- \tag{21}$$

Now our aim is to determine A using the boundary condition

$$\lim_{\eta\to-\infty}f'(\eta)=0 \tag{22}$$

Applying the boundary condition (22) to [2/2] Pade approximant of the derivative of the polynomial solution (21) we get,

$$\lim_{\eta\to-\infty}\frac{1+\frac{3}{4}A\eta+\frac{-3A^2+1}{12}\eta^2}{1-\frac{A}{4}\eta+\frac{1}{12}\eta^2}=0.$$

which gives $A=0.5773502693$.

Case II: ($M=1$)

The DTM expression (20) gives,

$$f(\eta)=\eta+\tfrac{A}{2}\eta^2+\tfrac{1}{6}\eta^3+\tfrac{A}{48}\eta^4-\tfrac{A^2}{240}\eta^5-\tfrac{A}{320}\eta^6-(\tfrac{8+5A^2}{20160})\eta^7+(\tfrac{8-71A+2A^3}{322560})\eta^8+--- \tag{23}$$

Applying the boundary condition (22) to [2/2] Pade approximant of the derivative of the polynomial solution (23) we get,

$$\lim_{\eta\to-\infty}\frac{1+\frac{8A^3-21A}{6(A^2-4)}\eta+\frac{16A^4+37A^2-96}{48(A^2-4)}\eta^2}{1+\frac{3A+2A^3}{6(A^2-4)}\eta-\frac{11A^2}{48(A^2-4)}\eta^2}=0$$

which gives $A=1.245963023848551$.

Case III: ($M=2$)

The DTM expression (20) gives,

$$f(\eta)=\eta+\tfrac{A}{2}\eta^2+\tfrac{1}{3}\eta^3+\tfrac{A}{16}\eta^4-\tfrac{A^2-4}{240}\eta^5-\tfrac{13A}{2880}\eta^6-(\tfrac{21A^2+32}{20160})\eta^7+(\tfrac{64-439A+2A^3}{322560})\eta^8+--- \tag{24}$$

Applying the boundary condition (22) to [2/2] Pade approximant of the derivative of the polynomial solution (24) we get,

$$\lim_{\eta\to-\infty}\frac{1+\frac{13A^3-40A}{12(A^2-4)}\eta+\frac{A^4+16A^2-44}{12(A^2-4)}\eta^2}{1+\frac{A^3+8A}{12(A^2-4)}\eta+\frac{1-A^2}{3(A^2-4)}\eta^2}=0$$

which gives $A=1.585084287112327$

6. NUMERICAL PROCEDURE

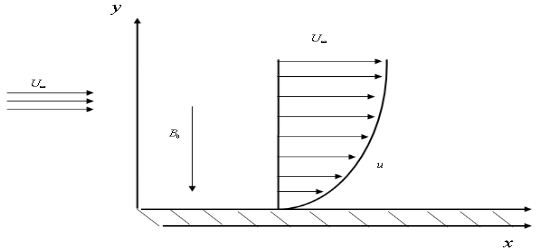

Figure A. Flow geometry

The non-linear differential equation (4) with the boundary conditions (5) and (17) constitute a boundary value problem which is solved numerically by the efficient shooting technique. The BVP is equivalent to a system of first order differential equations with boundary conditions. Equation (4) is integrated numerically by forth order Runge-Kutta scheme with the given boundary conditions and a guessed trail values of $f''(0)$. However the numerical solution thus obtained will not generally satisfy the right boundary condition. At this end Newton-Raphson scheme is employed to correct the arbitrary guess values such that the numerical solution will even satisfy the required boundary conditions (5) and (17). The convergence criterion largely depends on fairly good guesses of the initial conditions in the shooting technique. The iterative procedure is terminated until the relative difference between the current and the previous values are matches with a tolerance of 10^{-6}.

7. RESULTS AND DISCUSSION

An attempt has been made to solve classical Blasius boundary layer equation when the flow is subjected to a uniform magnetic field by differential transform method and Pade approximation.

Figure 1 presents the numerical solution of Blasius equation by DTM and DTM-Pade methods for both the cases; absence of magnetic field ($M = 0.0$) and presence of magnetic field ($M = 1.0$). The result is in good agreement with Peker et al. [12].

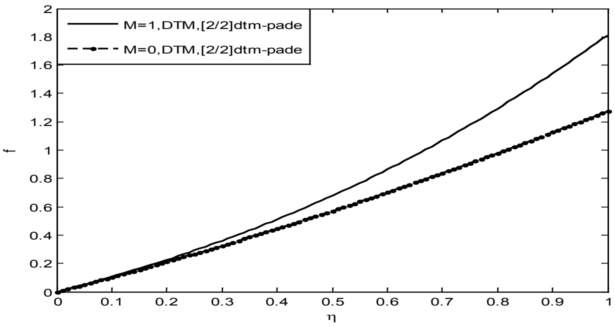

Figure 1. Results obtained by DTM and [2/2] dtm-pade

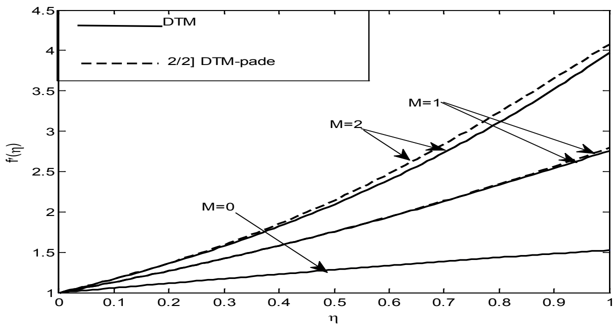

Figure 2. Comparison plot of f" by DTM and [2/2] dtm-pade

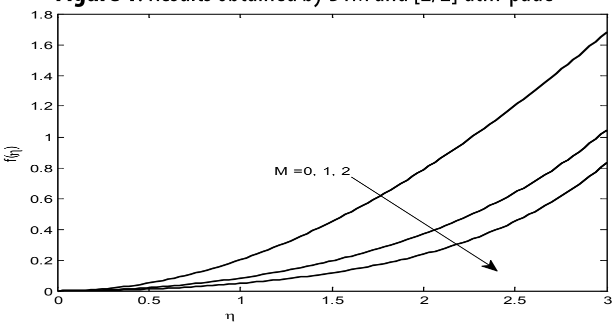

Figure 3. Transverse velocity (Numerical Methods)

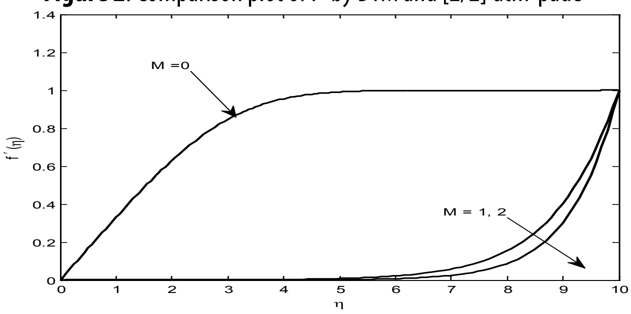

Figure 4. Longitudinal velocity (Numerical Method), presence of Kp

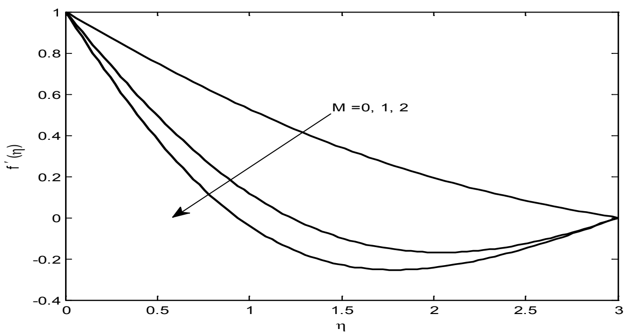

Figure 5. Longitudinal velocity (Numerical Method), absence of Kp

Figure 2 exhibits the velocity distribution of MHD flow under the influence of magnetic field for various values of magnetic parameter, M. It is interesting to observe that the solutions for both DTM and DTM associated with Pade, coincide (i.e. coincidence of dotted curve with continuous curve) in the absence of magnetic field ($M = 0.0$) and for small value of M i.e.($M = 1.0$). In case of $M = 2.0$, the solutions differ slightly but uniformly when $\eta > 0.35$. Therefore, it is suggested that higher order Pade approximants may yield better result for greater magnetic interaction. Further, it is observed that an increase in magnetic field contribute to the growth of boundary layer thickness.

The numerical approach to the transverse velocity is presented in the Figure 3 and comparison has been made for both the absence and presence of magnetic field. It is observed that the numerical investigation is an good agreement with that of DTM and DTM-Pade approximant solution. Hence it is conclude that Lorentz force has a retarding effect and the velocity field is asymptotic in nature with higher value of magnetic parameter. In the absence of magnetic parameter there is pick near the boundary layer.

Figure 4 and 5 both presents the longitudinal velocity profile in both the presence / absence of magnetic parameter. Figure 4 exhibits the velocity profile for the problem set (4) with the boundary condition (5) and Figure 5 presents the profile for the boundary condition (17) used for the classical Blasius flow [20].

Table-1 presents the values of skin friction for different values of the magnetic parameter by using both Pade approximant and numerical methods. The result is in good agreement with that of previous authors. It is observed that an increase in magnetic parameter leads to an increase in the values of skin friction. The same effect of magnetic parameter was observed by Su and Zheng [21].

Table-1: Skin friction coefficient

M	Skin Friction DTM-Pade	Skin Friction Blasius flow	Skin Friction Classical Blasius flow
0	0.577350269	0.5045587599	0.5078780101
1	1.245963023	1.1646497090	1.0638332069
2	1.585084287	1.3936314590	1.3308454067

8. CONCLUSION

It may be concluded that the present method is an efficient method to solve non-linear boundary layer equations. Moreover, inclusion of additional forcing forces such as magnetic field etc. needs higher order Pade approximants for the stability of the solutions.

References

[1.] J.-H. He, Homotopy perturbation technique, Computer Methods in Applied Mechanics and Engineering, 178 (3-4) (1999) 257–262.

[2.] M. Dehghan, F. Shakeri, Solution of an integro-differential equation arising in oscillating magnetic fields using He's homotopy perturbation method, Progress in Electromagnetic Research, 78, (2008) 361–376.

[3.] A. M. Lyapunov, The General Problem of the Stability of Motion, Taylor & Francis, London, UK, 1992, English translation.

[4.] M. Dehghan, F. Shakeri, The use of the decomposition procedure of Adomian for solving a delay differential equation arising in electrodynamics, Physica Scripta, 78, Article ID 065004, 11 pages, 2008.

[5.] A. V. Karmishin, A. I. Zhukov, V. G. Kolosov, Methods of Dynamics Calculation and Testing for Thin-Walled Structures, Mashinostroyenie, Moscow, Russia, 1990.

[6.] M. Dehghan, M. Shakourifar, A. Hamidi, The solution of linear and nonlinear systems of Volterra functional equations using Adomian-Pade technique, Chaos, Solitons and Fractals, 39,(5) (2009) 2509–2521.

[7.] J. H. He, A new approach to non-linear partial differential equations, Communications in Nonlinear Science and Numerical Simulation, 2, (4) 1997.

[8.] M. M. Rashidi, H. Shahmohamadi, Analytical solution of three-dimensional Navier–Stokes equations for the flow near an infinite rotating disk, Communications in Nonlinear Science and Numerical Simulation, 14, (7) (2009)2999–3006.

[9.] T. Hayat, C. Fetecau, M. Sajid, Analytic solution for MHD transient rotating flow of a second grade fluid in a porous space, Nonlinear Analysis: Real World Applications, 9 (4) (2008) 1619–1627.

[10.] T. Hayat, T. Javed, M. Sajid, Analytic solution for MHD rotating flow of a second grade fluid over a shrinking surface, Physics Letters A, 372 (18) (2008) 3264–3273.

[11.] M. M. Abdelkhalek, Heat and mass transfer in MHD flow by perturbation technique, Computational Materials Science, 43 (2008) 384–391.

[12.] H.A.Peker, O.Karaoglu, G.Oturanc, The differential transformation method and Pade approximant for a form of MHD Blasius flow, 16(2) (2011) 507-513.

[13.] J. K. Zhou, Differential Transformation and Its Applications for Electrical Circuits, Huazhong University Press, Wuhan, China, 1986.

[14.] M.-J. Jang, C.-L. Chen, Y.-C. Liu, On solving the initial value problems using the differential transformation method, 115 (2000) 145-160.

[15.] A.Arikoglu, I.Ozkol, Solution of difference equations by using differential transform method, Applied Mathematics and Computation, 174 (2006)1216–1228.

[16.] P.Darania, A.Ebadian, Amethod for the numerical solution of integro-differential equations, Applied Mathematics and Computation, 188 (2007) 657-668.

[17.] F. Ayaz, On the two-dimensional differential transform method, Applied Mathematics and Computation, 143, (2-3) (2003) 361–374, 2003.

[18.] A.Kurnaz, G.Oturanc, The differential transform approximation for system of ordinary differential equations, International journal of Computer Mathematics, 82 (2005) 709-719.

[19.] G.A.Baker, P.Graves-Marris, Pade Approximants, Cambridge U.P., 1996.

[20.] A.Wazwaz, The modified decomposition method and Pade approximants for a boundary layer equation in unbounded domain, Applied Mathematics and Computation, 177 (2006) 737-744.

[21.] X.Su and L.Zheng, Approximate solution to MHD Falkner-Skan flow over permeable wall, Appl. Math. Mech. Engl. Ed., 32(4) (2011) 401-408.

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