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THE METHOD OF LINES FOR SOLUTION OF THE TWO-DIMENSIONAL ELLIPTIC EQUATION

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ABSTACT: The solution of the two-dimensional elliptic equation is presented by the method of lines. The method of lines (MOL) is a general way of viewing a partial differential equation as a system of ordinary differential equations. The partial derivatives with respect to the one dimension spatial variable is discretized to obtain a system of ODEs in the second dimension spatial variable and then using analytic method with eigenvalues and eigenvectors to solve this ODEs system.
Keywords: Elliptic equation, Method of lines, Eigenvalues and Eigenvectors

1. INTRODUCTION

Over the last few years, it has become increasingly apparent that many physical phenomena can be described in terms of elliptic partial differential equations with the classical boundary condition [3,13,14].Growing attention is being paid to the development, analysis and implementation of numerical methods for the solution of these problems[1,2,3].Elliptic boundary value problems in two dimensions that have been studied by several authors [5 - 10].This equation have been solved directly by various numerical methods such as Adomain decomposition method, Finite volume method, Finite difference method, Finite element method and etc.

In this research, we consider the following problem of this family of equations

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0, \quad u=u(x,y), 0\le x\le a, 0\le y\le b \tag{1}$$

with Direchlet boundary conditions

$$u(0,y)=f_1(y) \tag{2}$$

$$u(a,y)=f_2(y) \tag{3}$$

$$u(x,0)=g_1(x) \tag{4}$$

$$u(x,b)=g_2(x) \tag{5}$$

where $f_1(y), f_2(y), g_1(x)$ and $g_2(x)$ are known functions. We assume that the functions $f_1(y), f_2(y), g_1(x)$ and $g_2(x)$ satisfy the conditions in order that the solution of this equation exists and is unique.

In this work a different approach is used, the solution of the above equation ((1)-(5)) is computed by semi analytical method of lines. Method of lines is an alternative computational technique which involves making an approximation to the one dimension space derivatives and by reducing the problem to that of solving a system of initial value ordinary differential equations to second dimension and then using a eigenvalues and its corresponding eigenvectors for solving the ODE system in analytical way.

This work is organized in the following way. In Section II, we introduce the method of lines briefly and apply it to the equations ((1)–(5)), In section III, we solve standard elliptic problem, some results and comparison with the numerical finite difference method and analytical method presented are given in section IV and finally a conclusion is drawn in section V.

2. METHOD OF LINES

Method of lines is a semi-discretized method [4,5,7,13,14] which involves reducing boundary value problem to a system of ordinary differential equations (ODEs) in time through the use of a discretization in one dimension space and second dimension space derivative convert in ODEs. The resulting ODEs system can be solved using the eigenvalues of coefficient matrix of the system, which may use a variable time-step/variable order approach with time local error control. The most important advantage of the MOL approach is that it has not only the simplicity of the explicit methods [13] but also the superiority (stability advantage) of the implicit ones unless a poor numerical method for solution of ODEs is employed. It is possible to achieve higher-order approximations in the discretization of spatial derivatives without significant increases in the computational complexity. This technique has the broad applicability to physical and chemical systems modeled by PDEs.

In order to use this approach for solving equations ((1)–(5)), the first step in our solution process is to replace $\frac{\partial^2 u}{\partial x^2}$ in Equation (1) by a finite difference approximation accurate to order h^2 such as

$$\frac{\partial^2 u}{\partial x^2} = \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} \tag{6}$$

where h is the spacing between discretized line.

The region is divided into the strips by N dividing straight lines (hence the name method of lines) parallel to the y direction.

$$h = \frac{a}{N+1} \tag{7}$$

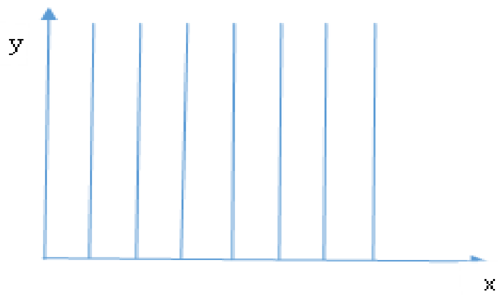

Figure 1. The approximations $u_i(y)$ are defined along the lines

Therefore equation (1) becomes

$$\frac{d^2 u_i}{dy^2} = -\left(\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2}\right), i = 1,2,\ldots,N \tag{8}$$

Also the boundary condition at $x = 0$ and $x = a$ (Equations (2) and (3)) are transformed as follows.

$$u_0 = f_1(y) \tag{9}$$

$$u_{N+1} = f_2(y) \tag{10}$$

and the boundary condition at $y = 0$ and $y = b$ (Equations (4) and (5)) are transformed as follows.

$$u_i\,|_{y=0} = g_1(ih), i = 1,2,\ldots,N \tag{11}$$

$$u_i\,|_{y=b} = g_2(ih), i = 1,2,\ldots,N \tag{12}$$

Here the governing equation (8) is second order system of ordinary differential equations. To solve this, convert in first order system of ordinary differential equations in the following manner.

$$\frac{du_i}{dy} = \frac{u_{N+1+i}}{h}, i = 1,2,\ldots,N \tag{13}$$

$$\frac{du_{N+1+i}}{dy} = \frac{-u_{i+1} + 2u_i - u_{i-1}}{h}, i = 1,2,\ldots,N \tag{14}$$

Equations ((13)-(14)) are system of $2N$ linear first order differential equations and can be written in matrix form as

$$\frac{dU}{dy} = AU + b \tag{15}$$

where, A is an $2N \times 2N$ coefficient matrix of $2N \times 1$ column unknown matrix U given by

$$A=\frac{1}{h}\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 1 \\ 2 & -1 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \tag{16}$$

$$U=[u_1 \quad u_2 \quad \cdots \quad u_N \quad u_{N+2} \quad \cdots \quad u_{2N+1}]' \tag{17}$$

and b is a column vector of order $2N\times 1$ which is in the form

$$b=\frac{1}{h}[0 \quad \cdots \quad 0 \quad u_0 \quad 0 \quad \cdots \quad 0 \quad u_{N+1}]' \tag{18}$$

The next step is to solve the equation (15) analytically along the y coordinate. Equation (15) have some basic steps to solve it.
If $b=0$ in equation (15) then it is called homogeneous system, so that it is

$$\frac{dU}{dy}=AU \tag{19}$$

To solve this, consider A has a basis of $2N$ eigenvectors $X^{(1)}, X^{(2)},..., X^{(2N)}$ corresponding to eigenvalues $\lambda_1, \lambda_2,..., \lambda_{2N}$ then the corresponding solutions of (19) are

$$u^{(1)}=X^{(1)}e^{\lambda_1 y}, u^{(2)}=X^{(2)}e^{\lambda_2 y},..., u^{(2N)}=X^{(2N)}e^{\lambda_{2N} y} \tag{20}$$

and the general solution is

$$U=c_1X^{(1)}e^{\lambda_1 y}+c_2X^{(2)}e^{\lambda_2 y}+...+c_{2N}X^{(2N)}e^{\lambda_{2N} y} \tag{21}$$

Here $c_1, c_2,..., c_{2N}$ are arbitrary constants.
Now, if $b\neq 0$ in equation (15) then it is called Non-homogeneous system and its solution is

$$U=U^{(h)}+U^{(p)} \tag{22}$$

Here $U^{(h)}$ is a general solution of the system (19) and $U^{(p)}$ is particular solution which calculate by method of undetermined coefficient or the method of the variation of parameters [15].

3. STANDARD HEAT TRANSFER PROBLEM

Consider the elliptic equation (Laplace equation) with Dirichlet boundary condition is

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \tag{23}$$

$$u(0,y)=0,\ 0\le y\le 1 \tag{24}$$

$$u(1,y)=0,\ 0\le y\le 1 \tag{25}$$

$$u(x,0)=0,\ \ 0\le x\le 1 \tag{26}$$

$$u(x,1)=\sinh(\pi)\sin(\pi x),\ \ 0\le x\le 1 \tag{27}$$

SOLUTION:
For simplicity, consider $N=3$ and using the above procedure of the method of lines.
We have $a=1\ \&\ b=1$.
Now, equations ((13)-(14)) becomes

$$\frac{du_i}{dy}=4u_{4+i}, i=1,2,3. \tag{28}$$

$$\frac{du_{4+i}}{dy}=4\left(-u_{i+1}+2u_i-u_{i-1}\right), i=1,2,3. \tag{29}$$

With

$$u_0=0 \tag{30}$$

$$u_4=0 \tag{31}$$

$$u_i\,|_{y=0}=0,\ i=1,2,3. \tag{32}$$

$$u_i\,|_{y=1} = \sinh(\pi)\sin(ih\pi),\ i = 1,2,3. \tag{33}$$

Therefore the matrix form of the system is

$$\frac{dU}{dt} = AU + b \tag{34}$$

where, A is an 6×6 coefficient matrix of U given by

$$A = \begin{bmatrix} 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 \\ 8 & -4 & 0 & 0 & 0 & 0 \\ -4 & 8 & -4 & 0 & 0 & 0 \\ 0 & -4 & 8 & 0 & 0 & 0 \end{bmatrix} \tag{35}$$

$$U = [u_1 \quad u_2 \quad u_3 \quad u_5 \quad u_6 \quad u_7]' \tag{36}$$

and b is a column vector of order 6×1 which is in the form

$$b = [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]' \tag{37}$$

Here b is zero vector because $u_0 = u_4 = 0$ such that system becomes homogeneous.

Now, The Eigen values and corresponding Eigen vectors of matrix A are

$\lambda_1 = -7.391036260090299$

$$X^{(1)} = \begin{bmatrix} 0.2379815744738984 \\ -0.336556770590778 \\ 0.237981574738984 \\ -0.4397321612032304 \\ 0.621875823753832 \\ -0.439732612032305 \end{bmatrix}$$

$\lambda_2 = -5.656854249492384$

$$X^{(2)} = \begin{bmatrix} -0.408248290463864 \\ 0 \\ 0.408248290463862 \\ 0.577350269189627 \\ 0 \\ -0.577350269189625 \end{bmatrix}$$

$\lambda_3 = -3.061467458920711$

$$X^{(3)} = \begin{bmatrix} -0.397052243880408 \\ -0.561516668266344 \\ -0.397052243880410 \\ 0.303890631032831 \\ 0.429766251884748 \\ 0.303890631032832 \end{bmatrix}$$

$\lambda_4 = 7.391036260090296$

$$X^{(4)} = \begin{bmatrix} 0.237981574738984 \\ -0.336556770590777 \\ 0.237981574738984 \\ 0.439732612032304 \\ -0.621875823753832 \\ 0.439732612032305 \end{bmatrix}$$

$\lambda_5 = 5.656854249492382$

$$X^{(5)} = \begin{bmatrix} -0.408248290463863 \\ 0 \\ 0.408248290463862 \\ -0.577350269189626 \\ 0 \\ 0.577350269189626 \end{bmatrix}$$

$\lambda_6 = 3.061467458920718$

$$X^{(6)} = \begin{bmatrix} 0.397052243880409 \\ 0.561516668266344 \\ 0.397052243880409 \\ 0.303890631032832 \\ 0.429766251884748 \\ 0.303890631032831 \end{bmatrix}$$

Thus, the general solution of ODEs system is

$$U = c_1X^{(1)}e^{\lambda_1 t} + c_2X^{(2)}e^{\lambda_2 t} + c_3X^{(3)}e^{\lambda_3 t} + c_4X^{(4)}e^{\lambda_4 t} + c_5X^{(5)}e^{\lambda_5 t} + c_6X^{(6)}e^{\lambda_6 t} \tag{38}$$

We call this is a semi analytical solution, $c_i\,(i = 1,2,3,4,5,6)$ are unknowns which calculate by using condition of equations (32) and (33) with some standard linear algebra method to solve system of linear equations.

So, the solution of interior node points are

$$\begin{aligned} u_1 &= 0.766344734780988\sinh(\lambda_6 y) \\ u_2 &= 1.083775117380485\sinh(\lambda_6 y) \\ u_3 &= 0.766344734780988\sinh(\lambda_6 y) \end{aligned} \tag{39}$$

Table 1:Analytic Solution (h = 0.25)

Y=	x= 0	0.25	0.50	0.75	1
0	0	0	0	0	0
0.25	0	0.614243127486596	0.868670961486010	0.614243127486596	0
0.50	0	1.627264059358646	2.301298902307295	1.627264059358646	0
0.75	0	3.696734399792561	5.227971924677803	3.696734399792561	0
1	0	8.166191913672924	11.548739357257748	8.166191913672925	0

Table 2:Numerical Solution (h = 0.05)

Y=	x= 0	0.25	0.50	0.75	1
0	0	0	0	0	0
0.25	0	0.247842784096184	0.350502626605130	0.247842784096184	0
0.50	0	0.846026680139686	1.196462405183028	0.846026680139686	0
0.75	0	2.640121641087168	3.733695831140186	2.640121641087168	0
1	0	8.166191913672924	11.548739357257748	8.166191913672925	0

Table 3:Method of Lines Solution(h = 0.25)

Y=	x= 0	0.25	0.50	0.75	1
0	0	0	0	0	0
0.25	0	0.645499703984623	0.912874435882871	0.645499703984623	0
0.50	0	1.687946970413908	2.387117498125925	1.687946970413908	0
0.75	0	3.768390739871204	5.329309292647036	3.768390739871204	0
1	0	8.166191913672943	11.548739357257768	8.166191913672943	0

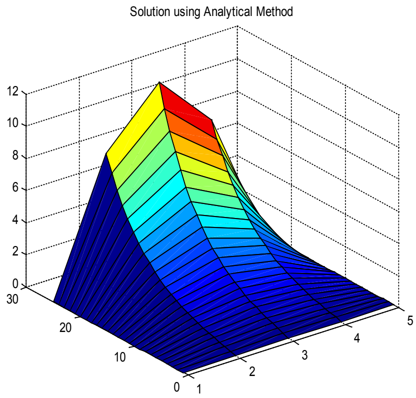

Figure 1: Analytic solution

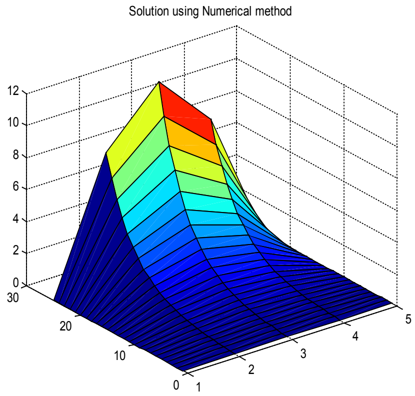

Figure 2: Numerical Solution

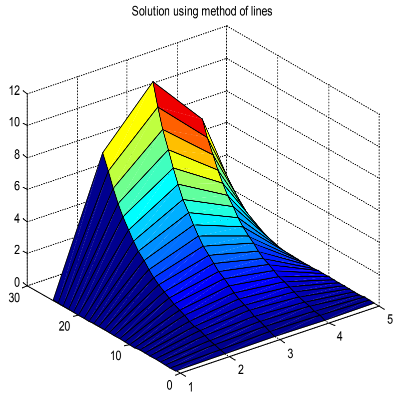

Figure 3: MOL solution

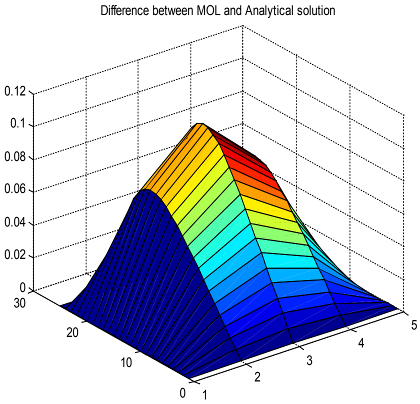

Figure 4: Difference between MOL and Analytic method

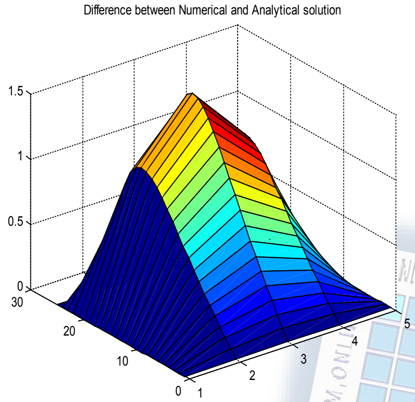

Figure 5: Difference between Numerical and Analytic method

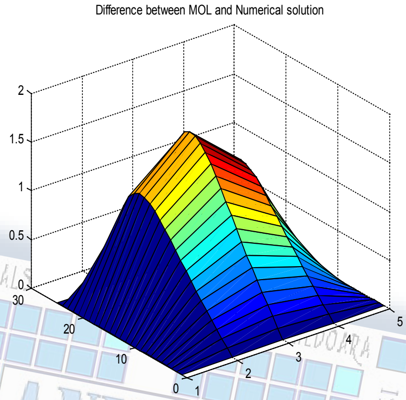

Figure 6: Difference between MOL and Numerical method

The MOL method usually enables us to solve quite general and complicated partial differential equations relatively easily and with acceptable accuracy. It is applicable to a wide range of problems in many areas.

4. CONCLUSION

The method of lines is generally recognized as a comprehensive and powerful approach to the numerical solution of elliptic equation. This method proceeds in two separate steps: Spatial derivatives are first replaced with finite difference or other algebraic approximations and then the resulting system of ordinary differential equations which is usually stiff, is integrated in another spatial variable domain. The work is provides the comparison between the analytical, numerical and method of lines solution and give the better idea that method of lines is quite accurate than the traditional numerical finite difference method.

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