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THERMOELASTIC ANALYSIS OF FUNCTIONALLY GRADED INCOMPRESSIBLE SPHERICAL BODIES

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ABSTACT: The main objective of this paper is to develop a method to deal with the thermo-mechanical analysis of incompressible, functionally graded hollow spherical bodies. The considered spherical components are subjected to combined thermal and mechanical loads and the equations of the displacement and stress fields are derived. The computations are executed when the distribution of material properties are given as power functions of the radial coordinate. The developed solutions are verified by finite element simulations.
Keywords: functionally graded, sphere, incompressible, thermal stresses

1. INTRODUCTION

Functionally graded materials are relatively new, advanced materials in which the material composition and parameters continuously vary with position. The gradual change between the material phases differentiates the behavior of these materials from the behavior of homogeneous and traditional composite materials. Functionally graded materials possess a lot of advantages which make them ideal choices in many engineering problems, for example space structures, cutting tools, furnace liners or fusion reactors. These materials have excellent mechanical properties, they are heat resistant and they take advantage of certain desirable features of each of the constituent phases.

The analytical solutions for the stress and displacement fields within functionally graded spherical bodies and circular cylinders are given by Lutz and Zimmerman in [1], [2]. Their papers considered spherical and cylindrical bodies under radial thermal load with linear gradient. The work of Tutuncu and Ozturk [3] gives closed-form analytical solutions for the stress field in functionally graded spherical bodies in case of internal pressure alone.

A study by Bayat, Mahdi and Torabi [4] presents an analytical solution to obtain the normal and effective stresses within thick spherical pressure vessels made of functionally graded materials subjected to axisymmetric mechanical and thermal loading. The properties of the material of the vessel are assumed to be graded in the radial direction based on power-law functions of the radial coordinate but the Poisson ratio has constant value.

In this paper the solution for a thermoelastic problem of a functionally graded hollow sphere is presented which works in the case of incompressible materials. In the last section we will further investigate the case, when the material parameters and the temperature field are power functions of the radial coordinate.

A thick spherical vessel will be considered in $Or\varphi\vartheta$ spherical coordinate system as we can see in Figure 1. The inner radius is denoted by R_1, the outer radius is R_2. The spherical body is radially graded, therefore the material properties are vary along the radial coordinate r. The thermal loading is a steady-state temperature difference field $T=T(r)=t(r)-t_0$ where t is the absolute temperature and t_0 is the reference temperature at which the stresses are zero if the spherical body is undeformed and the mechanical loads p_1 and p_2 are constant pressures exerted on the inner and outer boundary surfaces.

Our aim is to derive a method to calculate the displacement field and normal stresses within an incompressible spherical body.

2. FORMULATION OF THE PROBLEM

In case of incompressible materials the Poisson ratio ν=0.5 and for the Young modulus E=3G. The stress-strain relations for spherical bodies of the mechanical loading have the following forms [5], [6]

$$\sigma_r = 2G\varepsilon_r^m + \sigma_0 = \frac{2}{3}E\varepsilon_r^m + \sigma_0, \tag{1}$$

$$\sigma_\vartheta = 2G\varepsilon_\vartheta^m + \sigma_0 = \frac{2}{3}E\varepsilon_\vartheta^m + \sigma_0, \tag{2}$$

$$\sigma_\varphi = 2G\varepsilon_\varphi^m + \sigma_0 = \frac{2}{3}E\varepsilon_\varphi^m + \sigma_0, \tag{3}$$

where σ_r, σ_φ, σ_ϑ are the radial and tangential normal stresses, $\varepsilon^m{}_r$, $\varepsilon^m{}_\varphi$, $\varepsilon^m{}_\vartheta$ denote the normal strains from mechanical loads and

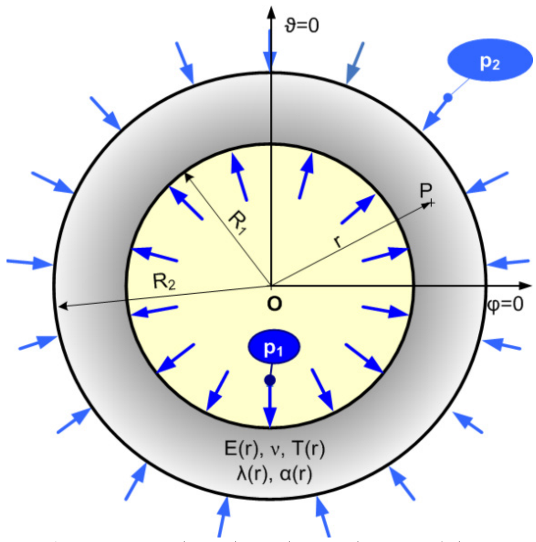

Figure 1. The sketch of the problem

$$\sigma_0 = \frac{1}{3}\left(\sigma_r + \sigma_\varphi + \sigma_\vartheta\right) = \frac{1}{3}\left(\sigma_r + 2\sigma_\varphi\right). \tag{4}$$

The normal strains can be written as the sum of its mechanical and thermal parts

$$\varepsilon_r(r) = \varepsilon_r^m(r) + \varepsilon_r^T(r) = \frac{\sigma_r(r) - \sigma_\varphi(r)}{E(r)} + \alpha(r)T(r), \tag{5}$$

$$\varepsilon_\varphi(r) = \varepsilon_\varphi^m(r) + \varepsilon_\varphi^T(r) = \frac{-\sigma_r(r) + \sigma_\varphi(r)}{2E(r)} + \alpha(r)T(r). \tag{6}$$

For the trace of the strain tensor the following relation can be written

$$\varepsilon = \varepsilon_r + \varepsilon_\varphi + \varepsilon_\vartheta = \varepsilon_r + 2\varepsilon_\varphi = 3\alpha T \tag{7}$$

The displacement-strain relations of spherical bodies are

$$\varepsilon_r(r) = \frac{du(r)}{dr}, \quad \varepsilon_\varphi(r) = \varepsilon_\vartheta(r) = \frac{u(r)}{r}. \tag{8}$$

The combination of Eqs. (7-8) leads to

$$\varepsilon(r) = \frac{du(r)}{dr} + 2\frac{u(r)}{r} = \frac{1}{r^2}\frac{d}{dr}\left(u(r)r^2\right) \tag{9}$$

The solution of Eqs. (7), (9) gives the function of radial displacement field

$$u(r) = \frac{3\int_{R_1}^{r} \rho^2\alpha(\rho)T(\rho)d\rho}{r^2} + \frac{C_1}{r^2} = 3\frac{F_1(r)}{r^2} + \frac{C_1}{r^2}, \tag{10}$$

where the following notation is introduced

$$F_1(r) = \int_{R_1}^{r} \rho^2\alpha(\rho)T(\rho)d\rho. \tag{11}$$

The substitution of Eq. (10) to the expressions of the normal strains Eqs. (5), (6) leads to the following formulae:

$$\varepsilon(r) = 3\alpha(r)T(r) - 6\frac{F_1(r)}{r^3} - 2\frac{C_1}{r^3}, \tag{12}$$

$$\varepsilon_\varphi(r) = \varepsilon_\vartheta(r) = 3\frac{F_1(r)}{r^3} + \frac{C_1}{r^3}. \tag{13}$$

In the case of hollow spherical bodies the equilibrium equation can be expressed as

$$\frac{d\sigma_r}{dr} = 2\frac{\sigma_\varphi - \sigma_r}{r}. \tag{14}$$

The combination of Eqs. (14) with Eqs. (1-2, 12-13) leads to

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$$\frac{d\sigma_r}{dr} = 2\left(6\frac{E(r)F_1(r)}{r^4} + 2\frac{E(r)}{r^4}C_1 - 2\frac{E(r)\alpha(r)T(r)}{r}\right). \tag{15}$$

The solution of Eq. (15) gives the function of the radial normal stress

$$\sigma_r(r) = 12\int_{R_1}^{r}\frac{E(\rho)F_1(\rho)}{\rho^4}d\rho + 4C_1\int_{R_1}^{r}\frac{E(\rho)}{\rho^4}d\rho - 4\int_{R_1}^{r}\frac{E(\rho)\alpha(\rho)T(\rho)}{\rho}d\rho + C_2. \tag{16}$$

The unknown constants C_1 and C_2 can be calculated from the stress boundary conditions:

$$p_1 = -\sigma_r(R_1),\;\; p_2 = -\sigma_r(R_2), \tag{17}$$

Form Eq. (17) it follows that

$$C_2 = \sigma_r(R_1), \tag{18}$$

$$C_1 = \frac{\sigma_r(R_2) - \sigma_r(R_1) - 12\int_{R_1}^{R_2}\frac{E(\rho)F_1(\rho)}{\rho^4} + \frac{E(\rho)\alpha(\rho)T(\rho)}{3\rho}d\rho}{4\int_{R_1}^{R_2}\frac{E(\rho)}{\rho^4}d\rho}. \tag{19}$$

The tangential normal stresses can be calculated from Eqs. (14-16)

$$\begin{aligned}\sigma_\varphi(r) &= \frac{1}{2}r\frac{d\sigma_r}{dr} + \sigma_r = \\ &= 2E(r)\left(3\frac{F_1(r)}{r^3} + \frac{C_1}{r^3} - \alpha(r)T(r)\right) + C_2 + \\ &+ 4\int_{R_1}^{r}E(\rho)\left(3\frac{F_1(\rho)}{\rho^4} + \frac{C_1}{\rho^4} - \frac{\alpha(\rho)T(\rho)}{\rho}\right)d\rho.\end{aligned} \tag{20}$$

3. EXAMPLE

For the numerical example the following functions will be used to describe the distribution of the material properties and temperature field within the functionally graded sphere:

$$\alpha(r) = \alpha_0\left(\frac{r}{R_1}\right)^{m_1}, E(r) = E_0\left(\frac{r}{R_1}\right)^{m_2},\;\; T(r) = T_0 r^{m_3}. \tag{21}$$

The following data will be used for the computations:

$$R_1 = 0.5m, R_2 = 0.7m, \alpha_0 = 1.2\cdot 10^{-6}\frac{1}{K}, E_0 = 210GPa,$$

$$T_0 = 1494\frac{K}{m^{m_3}}, m_1 = m_2 = m, m_3 = 5.638, p_1 = 30MPa, p_2 = 0.$$

Figure 2 shows the curves of the radial normal stresses by three different values of power indexes $m_1=m_2=m=(0, 0.2, 2)$. The tangential normal stresses can be seen in Figure 3.

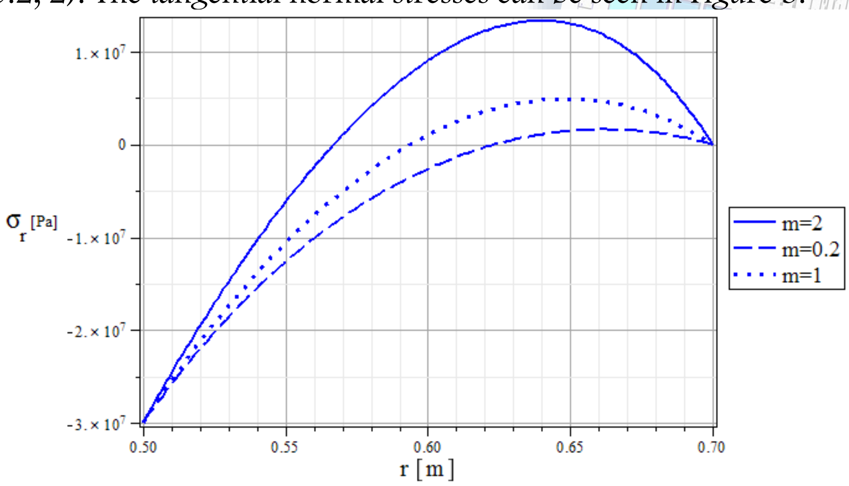

Figure 2. The plots of the radial normal stresses

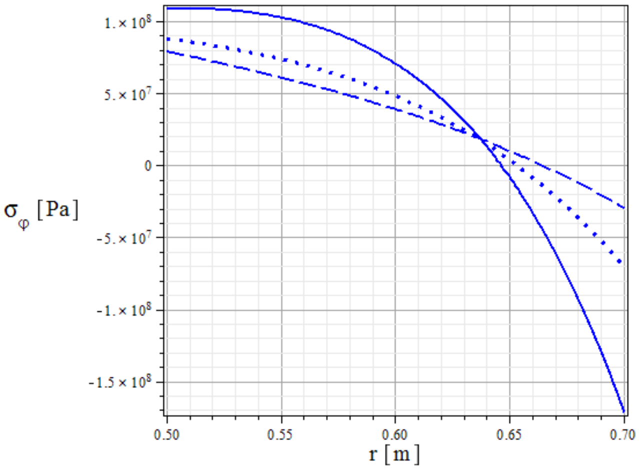

Figure 3. The plots of the tangential normal stresses

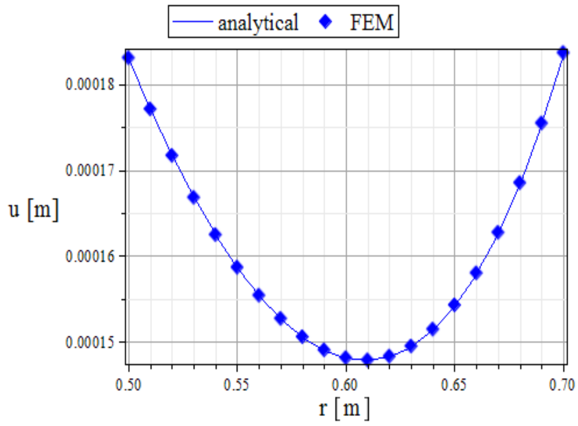

Figure 4. The comparison of the results for displacement field when m=2.

Next the solutions will be compared with finite element simulation. In the FE model the axisymmetric functionally graded sphere is modeled as a multilayered body with n=20 homogeneous spherical layers, as presented in [7]. In this case the displacement field can be seen in Figure 4. The results for the normal stresses are identical to the previously presented plots.

4. CONCLUSIONS

A solution was presented to determine the normal stresses and displacements for a thermoelastic problem of incompressible, functionally graded hollow spherical bodies subjected to thermal and mechanical loads. Numerical examples are presented when the distribution of material parameters are given as power functions of the radial coordinate. The developed solutions are verified by finite element models.

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