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PARAMETERIZATION OF THE ELLIPSE BASED ON THE VALENCIA'S SPHERE, WITHOUT TO USE A CARTESIAN COORDINATE SYSTEM

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Abstract: This article discusses a new geometric method for the parameterization of the ellipse based on the Dandelin theorem and the Valencia sphere, with the corresponding interpretation of the conic sections. This method for the determination of the ellipse's parameters, without to use a Cartesian coordinate system, applies basic trigonometric identities and right triangles. The concepts based on the Valencia's sphere can be implemented into robust mathematical models as a new pedagogical resource, in the subjects related to the study and analysis of the ellipses.
Keywords: conic sections, Dandelin theorem, parameterization of ellipse, polarity director plane, Valencia's sphere

1. INTRODUCTION

The conic sections are the non-degenerate curves generated by the intersections of a plane with one or two nappes of a cone. The conic sections have been studied by the ancient Greek mathematicians with this work culminating around 300 B.C., when Apollonius of Perga undertook a systematic study of their properties entitled *On Conics*. The three types of conic section are the hyperbola, the parabola, and the ellipse. The circle is a special case of the ellipse and it was sometimes called a fourth type of conic section. For a plane that is not perpendicular to the axis and that intersects only a single nappe, the curve produced is either an ellipse or a parabola. The curve produced by a plane intersecting both nappes is a hyperbola. The ellipse and hyperbola are known as central conics. There are many applications of conic sections in both pure and applied mathematics [1-3].

In geometry, the Dandelin spheres were discovered in 1822 by the Belgian mathematician Germinal Pierre Dandelin (1794 - 1847). The Dandelin spheres are one or two spheres that are tangent both to a plane and to a cone that intersects the plane. The intersection of the cone and the plane is a conic section, and the point at which either sphere touches the plane is a focus of the conic section, so the Dandelin spheres are also sometimes called focal spheres [3].

Pierce Morton (1803 - 1859) was an Irish mathematician who published a new proof of the focus-directrix property of conic sections using Dandelin spheres in the first volume of the Cambridge Philosophical Transactions (1829) [3]. The focus-directrix property is essential to proving that astronomical objects move along conic sections around the Sun [4].

A right cone can be generated by moving a line (the generatrix) fixed at the future apex of the cone along a closed curve (the directrix); if that directrix is a circle perpendicular to the line connecting its center to the apex, the motion is rotation around a fixed axis and the resulting shape is a circular cone [2].

If we take the intersection of a plane with a cone, the section so obtained is called a conic section [2, 5-7]. Thus, conic sections are the curves obtained by intersecting a right circular cone by a plane. We obtain different kinds of conic sections depending on the position of the intersecting plane with respect to the cone and the angle made by it with the vertical axis of the cone [8].

Let's consider the cutting plane angle α between the cutting plane and the horizontal plane, the conicity angle θ, and β is the angle between the vertical axis and the generatrix (fig. 1). The intersection of the plane with the cone can take place either at the vertex of the cone or at any other part of the nappe either below or above the vertex. When the plane cuts the nappe (other than the vertex) of the cone, we have the following situations:

a) when $\alpha = 0°$, the section is a circle.
b) when $0° < \alpha < \theta$, the section is an ellipse.
c) when $\alpha = \theta$; the section is a parabola. (In each of the above three cases, the plane cuts entirely across one nappe of the cone).
d) when $\theta < \alpha \leq 90$; the plane cuts through both the nappes and the curves of intersection is a hyperbola.

Various parameters are associated with a conic section. Major and minor axes of conics are the axes of symmetry for the figure 1.

The eccentricity is measured as the ratio of the distance between the two focal points and the major axis, $\varepsilon = c/a$. The eccentricity is a measure for how circle-like an ellipse is. An ellipse with an eccentricity of $\varepsilon = 0$ is just a circle. For $0 < \varepsilon < 1$ we obtain an ellipse, for $\varepsilon = 1$ a parabola, and for $\varepsilon > 1$ a hyperbola. Inbetween there's a continuum of possible shapes [3].

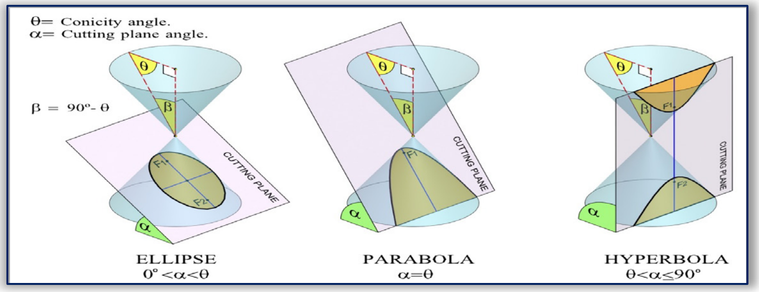

Figure 1. Conic sections

According to the study carried out by Dandelin and Quetelet [9], it was established that "*Any sphere tangent to the interior of the right cone and to the secant plane determines the focus of the conic section*", in addition, *"The circle of tangency common to the sphere and the right cone belongs to the plane that is perpendicular to the same axis; in other words the plane will intersect the secant plane in the directrix of the conic (D_1 or D_2); The obtained directrices will be perpendicular to the major axis of the ellipse, and parallel to the minor axis of the ellipse. "*

According to figure 2, the centers of the Dandelin spheres always belong to the axis of the right cone; and the shortest or perpendicular distance that starts from the center of each sphere until it intersects the cutting plane α, determines the exact position of the focus of the conic (F_1 and F_2) [10].

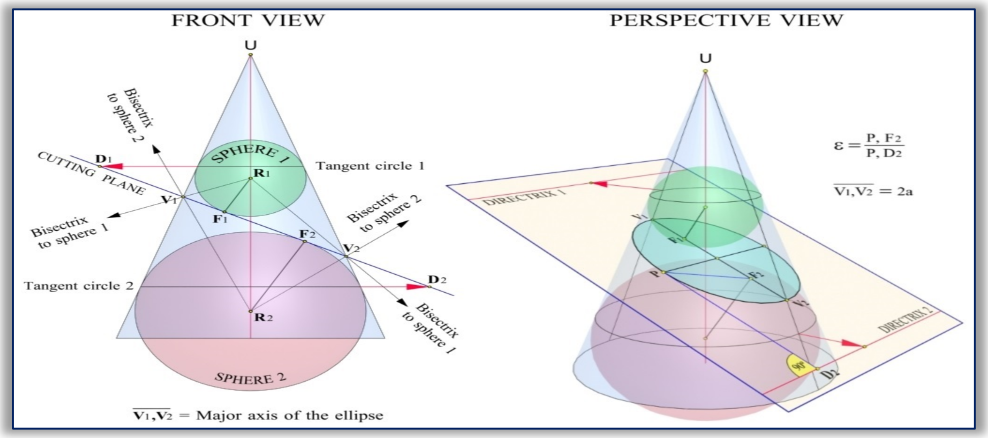

Figure 2. The components of an ellipse with the Dandelin spheres and the focus-directrix property

2. THE PROPOSED METHODOLOGY

The main objective is to determine the parameters of the ellipse, without to use a Cartesian coordinate system, based on the Valencia's sphere.

The intersection of a secant plane with a right circular cone with an angle of inclination, less than the conicity angle ($\alpha < \Theta$), generates an ellipse. The right circular cone has the height *H*. Let's note the point *L* (at the height *h*) where the secant plane intersects the vertical axis. This notation is not valid in the case of the hyperbola, when the angle $\alpha = 90^o$, since the secant plane is parallel to the axis of the right circular cone, and therefore, the point *L* can't be located.

Let's introduce the angle γ between the secant plane and the axis of the right circular cone. Applying the formula $\varepsilon = (\cos\gamma)/(\cos\beta)$, it can be seen the particular values of ε for every conic section, according to the cutting plane angle α, as shown in figure 3, [3].

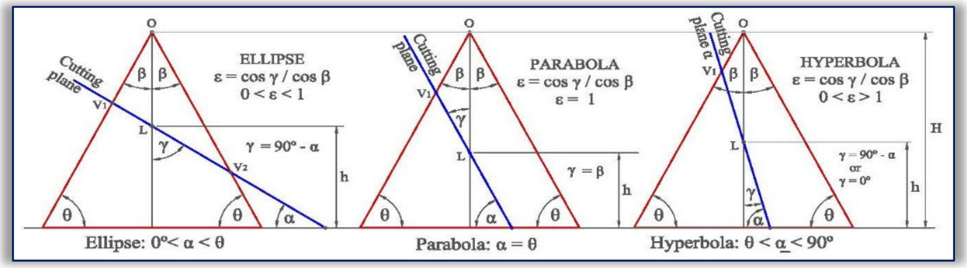

Figure 3. The main parameters used in geometrical constructions of conic sections

The main idea is to establish the radius of the Dandelin sphere as if it were an opposite cathetus; this being orthogonal to the secant plane, has its adjacent cathetus as the distance from a point *P* of the conic to the focus. "*The right triangle contained in Dandelin's sphere can be rotated on its opposite cathetus or focal radius, to construct the conic section*[1]*; in the case of the ellipse, the rotation is complete, defining a cone of variable revolution; while in the* *cases of parabola and hyperbola, the rotation is restricted because these are open curves that tend towards infinity*". ([1] Geometric constructions proposed by authors).

According to the previous geometric construction described, the values of the opposite and adjacent angles, as well as the lengths of the adjacent cathetus and the hypotenuse vary, when the rotation of the right triangle over its constant focal radius (*Rf*) is performed.

When the adjacent cathetus coincides with the major axis of the conic, the angle at the vertex *P*, may have the maximum or minimum allowed value, depending on whether the point *P* is located exactly at the point V_1 or V_2; therefore, this is the essential start point to establish the formulas for the determination of the conic's components, as shown in figure 4.

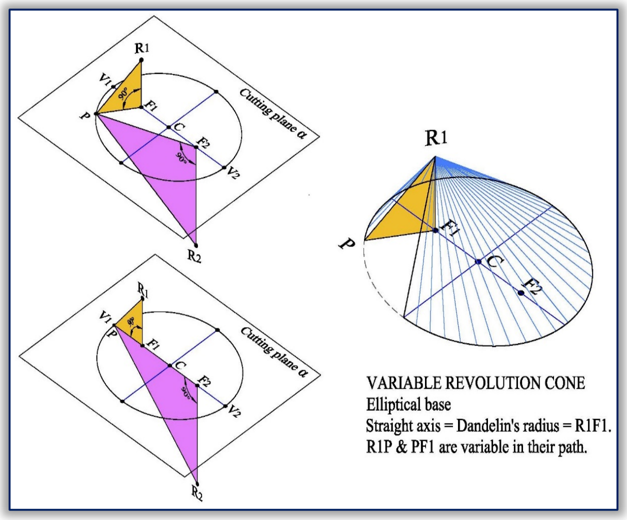

Figure 4. Right triangles orthogonal to the ellipse's plane

In figure 5, in the front view, there are shown the following geometrical elements: the cutting plane α displayed as an edge; the major axis of the conic; the right triangles contained in the Dandelin spheres; the tangential edges; and the axis of the right circular cone (parallel to the frontal plane), and consequently, their true lengths in projection [2]; thus, the frontal view resembles a drawing with characteristics of Euclidean geometry.

Let's apply the property of the external angle to a triangle for the "secant line", which traverses the two generatrices of the right circular cone [3]. The external angle located above the secant line is equal to the sum of the two internal angles not adjacent (angle = γ + β), and the external angle located below the secant line, is equal to the difference of the non-adjacent internal angles (angle = γ - β). Finally, the angles are specified supplementary and opposed by the vertex, as shown in figure 5.

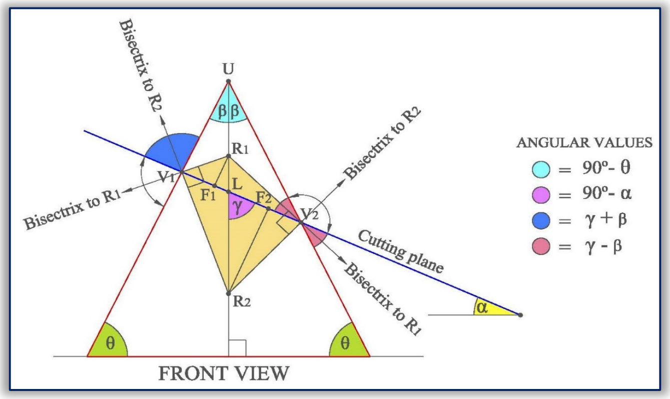

Figure 5. The front view of the right circular cone with its real and supplementary angles, and bisectrices

The bisectrices of the external angles are drawn by points V_1 and V_2, and intercept the axis of the right circular cone in the points R_1 and R_2, which coincide exactly with the centers of the Dandelin spheres. The perpendiculars drawn from R_1 and R_2 towards the secant plane α will determine the foci of the conic, complying with Dandelin's theorem [3], and will be called the focal radii: R_1F_1 and R_2F_2. The angles opposite the focal radii are adjacent and complementary angles to each other, as shown in figure 5.

The AutoCAD software allows users to model, analyze and design complex 2D and 3D structures with high precision [10-12].

All the geometrical aspects previously described, will be used to determine the ellipse's parameters in three-dimensional (3D) space, without to use a Cartesian coordinate system.

3. THE STUDY OF ELLIPSES

— The conical ellipse

An ellipse is a curve in a plane surrounding two focal points such that the sum of the distances to the two focal points is constant for every point on the curve [2, 3].

In descriptive geometry, it is necessary to construct an auxiliary projection plane adjacent to the elevation view, which is parallel to the secant plane α, to obtain the true size of the ellipse's projection; the focal radii will be parallel to the direction of the visual rays, and consequently, they will be projected as a point in F_1R_1 and R_2F_2; in this way, the dihedral projection of an ellipse resembles the points of view described by the mathematicians Dandelin and Quetelet, with the study of the Dandelin spheres in 1822 [10], (as shown in figure 6).

Let's consider a right circular cone, with height $H = 130$ and conicity angle $\Theta = 66º$, which is crossed by a secant plane with an inclination angle α = 36º. The intersection between the secant plane with the vertical axis determines a point L located at a height $h = 75$. Let's determine the parameters of the conic section resulted by the intersection of the secant plane with the right circular cone.

Since the angle of the secant plane is smaller than the conicity angle ($\alpha < \Theta$), it is obtained an ellipse. In front view is determined the length of the major axis of the ellipse ($V_1V_2 = 2a$), as shown in figure 7.

We proceed to find the angles complementary to the conicity angles Θ and the secant plane α:

≡ angle between the generatrix and cone axis: $\beta = 90° - \Theta$, that is $\beta = 90° - 66°$. So, $\beta = 24°$.

≡ angle between the axis of the cone and the secant plane: $\gamma = 90° - \alpha$, that is $\alpha = 90° - 36°$. So, $\gamma = 54°$.

≡ then are determined, the values of the external angles whose vertices coincide with V_1 and V_2.

≡ external angle above the secant plane: $\gamma + \beta = 54° + 24°$. So, $\gamma + \beta = 78°$.

≡ external angle below the secant plane: $\gamma - \beta = 54° - 24°$. So, $\gamma - \beta = 30°$.

≡ the difference of elevation $h' = H - h$, that is $h' = 130 - 75$. So, $h' = 55$.

According to the values found above, we proceed to determine the length of the major axis of the ellipse ($2a$). For this it is necessary to use the the law of sines [1, 4].

» Determination of the length of the major axis of the ellipse, V1V2

$$\frac{V_1L}{\sin\beta} = \frac{h'}{\sin[180° - (\beta+\gamma)]}, \text{ or}$$

$$V_1L = \frac{\sin\beta \cdot h'}{\sin[180° - (\beta+\gamma)]} \quad (1)$$

$$\frac{V_2L}{\sin\beta} = \frac{h'}{\sin(\gamma-\beta)}, \text{ or } V_2L = \frac{\sin\beta \cdot h'}{\sin(\gamma-\beta)} \quad (2)$$

By replacing the numerical values, there are obtained next values:

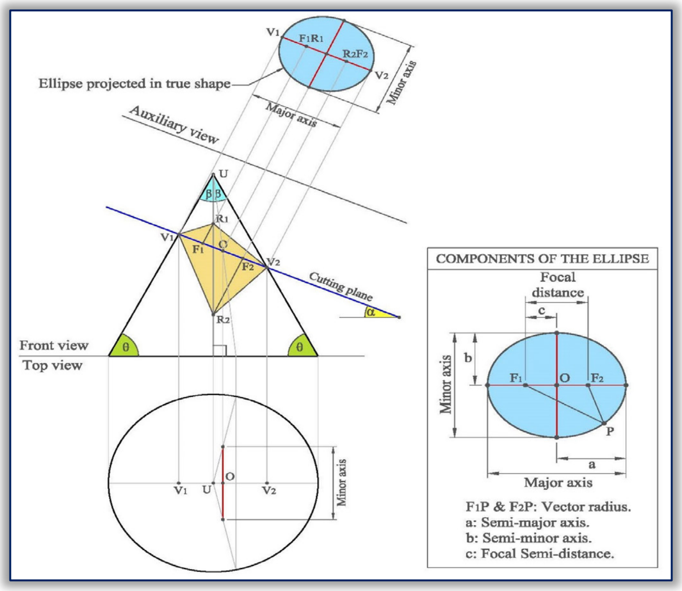

Figure 6. Projection of an ellipse in descriptive geometry and its components

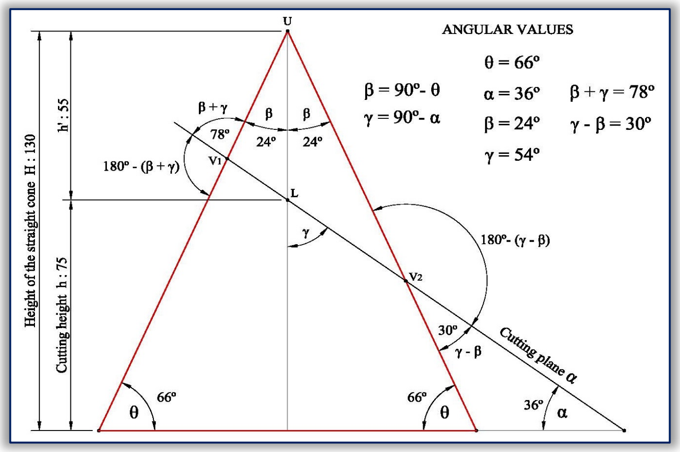

Figure 7. Geometrical constructions for a right circular cone with an elliptical section

$$V_1L = 22.87028583 \text{ and } V_2L = 44.7410304.$$

The major axis of the ellipse $2a = V_1V_2 = V_1L + V_2L$.

By replacing the numerical values, it is obtained: $2a = 67.61131623$. So, $a = 33.80565811$.

If the right circular cone is intersected by several secant planes parallel to each other, all the ellipses obtained will be similar, and therefore, would have the same eccentricity.

» The geometrical construction of the Valencia's sphere, which contains the ellipse

Definition proposed by authors: *The surface of the Valencia's sphere, always passes through the two vertices of the ellipse and the two centers of the Dandelin spheres, thus defining a remarkable quadrilateral of Ptolemy* [3]. Also, the centers of the Dandelin spheres are located exactly in each pole of the Valencia's sphere, and the vertices of the major axis of the ellipse define a chord, which are located with latitudes equal to the sum and difference of the angles of the secant plane and the angle between the generatrix and the axis of the right circular cone ($\alpha \pm \beta$). The Valencia's sphere represents a very simple and precise geometric method, useful to determine the parameters of the ellipse, starting from a known length of the major axis of the ellipse, and the angles of conicity and the secant plane, as shown in figure 8.

This geometric object complies with the following theorem:

The theorem of the sphere of Valencia: "The radius of the sphere of Valencia (Rva) is equal to the quotient that results from the length of the semi-major axis of the ellipse, and the cosine of the angle β between the generatrix and the axis of the right circular cone. The geometric center of the sphere of Valencia, is located below the secant plane, and the mediatrix of the major axis of the ellipse, is a distance equals with the product of the semi-major axis of the ellipse, and the tangent of the angle formed by the generatrix and the axis of the right circular cone (β), which determines the center O of the ellipse."

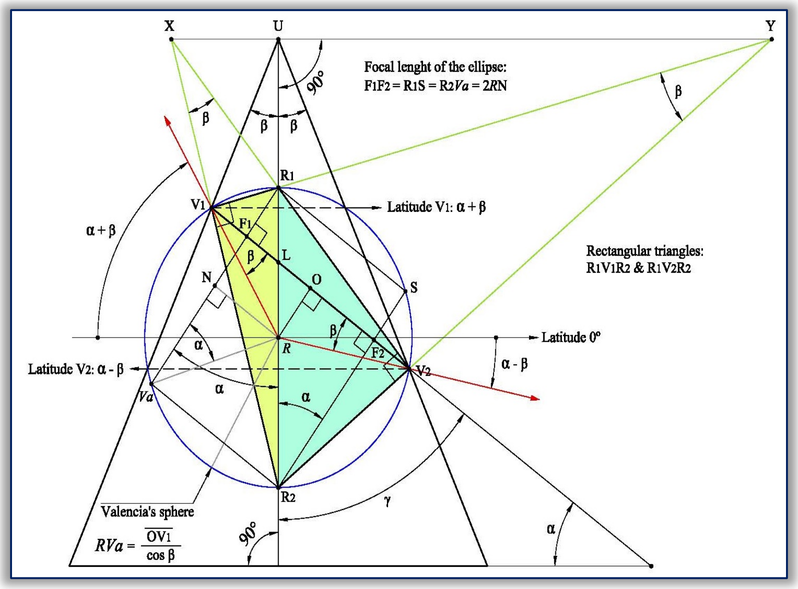

Figure 8. The sphere of Valencia

» Determination of the radius of the Valencia's sphere

The radius of the Valencia sphere *RVa* can be computed knowing the length of the semi-major axis of the ellipse, and the angle β between the generatrix and the vertical axis of the right circular cone, as:

$$\mathrm{RVa} = \frac{\mathrm{OV_1}}{\cos\beta} \tag{3}$$

By replacing the numerical values, it is obtained: *RVa* = 37.0049.

» Determination of the mediatrix of the ellipse, from the center of the Valencia's sphere

The geometric center *R* of the Valencia's sphere is situated on the vertical axis of the right circular cone, and is the midpoint between the two centers of the Dandelin spheres. The perpendicular *RO* on the major axis of the ellipse drawn from the center of the sphere of Valencia, is expressed as:

$$\mathrm{RO} = \mathrm{OV_1} \cdot \tan\beta \tag{4}$$

By replacing the numerical values, it is obtained: *RO* = 15.05124887.

The distance *RO*, can also be computed by applying the power of a circle with respect to the point [3], that is, the power of point *O*, with respect to the radius of Valencia's sphere.

$$\mathrm{RO} = \sqrt{(\mathrm{RVa})^2 - (\mathrm{OV_1})^2} \tag{5}$$

By replacing the numerical values, it is obtained: *RO* = 15.05124887.

» Determination of the focal semi-distance of the ellipse

The extension of the focal radius R_1F_1 determines a chord R_1Va. The mediatrix of this chord (*RN*), drawn from the center of the sphere of Valencia, is equal to the length of the focal semi-distance of the ellipse *c*:

$$\mathrm{RN} = \mathrm{RVa} \cdot \sin\alpha \tag{6}$$

By replacing the numerical values, it is obtained: *RN* = 21.750934.

» Determination of the minor semi-axis of the ellipse

The semi-minor axis (*b*) of an ellipse (*OB*), being perpendicular to the focal semi-distance *c* (OF_1) its length can be computed using the Pythagorean theorem [3].

The hypotenuse defined between these two cathetuses is equal to the length of the semi-major axis of the ellipse ($V_1O = V_2O = F_1B = F_2B$), as shown in figure 9.

$$\mathrm{OB} = \sqrt{(\mathrm{V_1O})^2 - (\mathrm{OF_1})^2} \tag{7}$$

By replacing the numerical values, it is obtained: *OB* = 25.878936672.

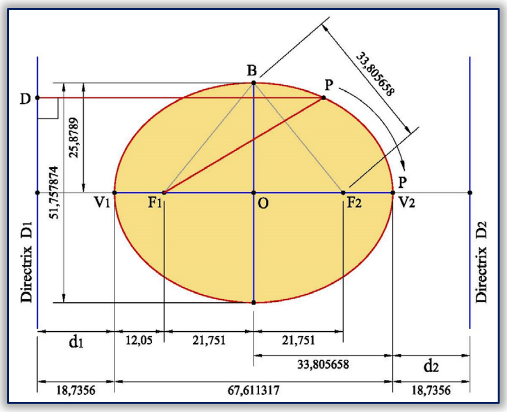

Figure 9. The numerical values of the computed ellipse

» Determination of the eccentricity and directrices of the ellipse

If a point *P* of the ellipse is situated on the major axis, exactly at the vertex farthest from the directrix (D_1), the maximum radius vector is obtained, equivalent to the sum of the focal axis with the focus-vertex distance.
If the eccentricity of the ellipse is the quotient between the focal semi-distance (*c*) and the semi-major axis (*a*), the following equation is obtained by equalization.

$$\varepsilon = \frac{c}{a}, \text{ or } \varepsilon = \frac{RN}{OV_1} = \frac{OF_1}{OV_1} \tag{8}$$

By replacing the numerical values, it is obtained: ε = 0.64341105.
If the point *P* is located in the vertex V_2, by equalization it is obtained:

$$\varepsilon = \frac{a+c}{2a+d} = \frac{c}{a} \tag{9}$$

The following relation is obtained by calculation:

$$d = \frac{a^2 - ac}{c} \tag{10}$$

By replacing the numerical values, it is obtained: *d* = 18.73564842.

» The notable quadrilateral of Ptolemy

The vertices of the ellipse and the centers of the Dandelin spheres determine a remarkable quadrilateral of Ptolemy, inscribed in the sphere of Valencia. The diagonals of the specified quadrilateral are: the major axis of the ellipse, and the diameter of the sphere comprised between the centers of the Dandelin spheres,
In this way, two right triangles $R_1V_1R_2$ and $R_1V_2R_2$ are defined, which share the same hypotenuse equivalent to the diameter of the sphere of Valencia, and that always passes through the polar axis of specified sphere. It can be noted that this polar diameter is collinear to the axis of the right circular cone; and consequently, this direction is fixed, but, its diameter is variable depending on the angular value of the secant plane.
Its importance lies in the fact that the internal angles of the vertices of the major axis of the ellipse are always right angles, since they comply with the second theorem of Thales [3]; by extending each pair of opposite sides, they coincide at a point that will always be located in the *polarity director plane*[3], analogous to *Monje's polar reciprocal theorem* [3].
The *polarity director plane* is a plane perpendicular to the axis of revolution, passing through the apex of the right cone (acronym: PDP), according to the authors.
The angles of convergence are equal to the angle β between the generatrix and the axis of the right circular cone; in this way, the two points of convergence *X, Y*, define a straight line that always passes through the vertex *U* of the right circular cone, and is coplanar to the polarity director plane PDP, as shown in figure 10.
According to the aspects described above, the vertices of the ellipse are located by means of latitudes equal to the sum and difference of the angles of the secant plane, and the angle between the generatrix and the axis of the right cone (α ± β). The prolongation of the bisector *RO* of the major axis of the ellipse, from the center of the Valencia's sphere, intersects the *XY* line in point *W*. According to the polar reciprocal theorem, the point *W* is in the polar of *O*, and vice versa.
This means that the polar line *RW* represents the axis of a cone of revolution. The cone of revolution is tangent to the Valencia's sphere, and its points of tangency coincide exactly with the vertices of the major axis of the ellipse. The polar circle of tangency has as geometric center the point *O*, which contains the ellipse inscribed in specified polar circle. The angle between a generatrix of this cone of revolution and the polar line *RW* is the same angle β of the right circular cone.

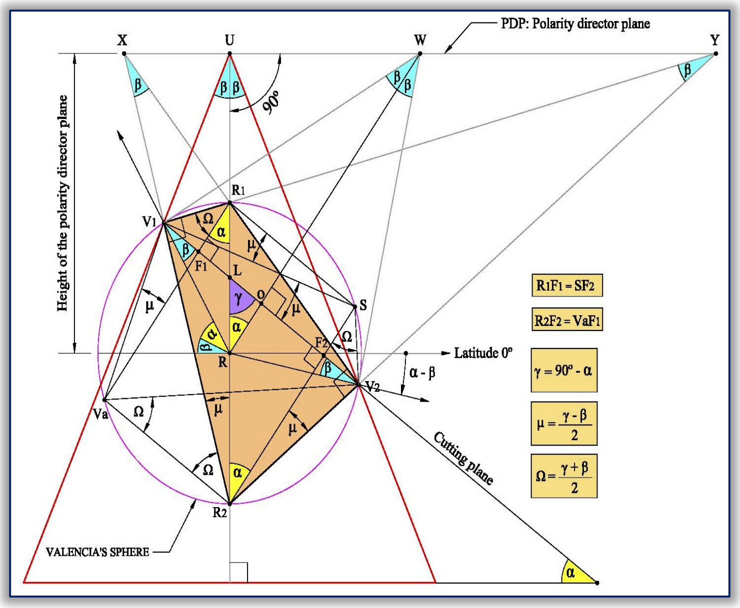

Figure 10. Gtaphical representations of convergence of the quadrilateral of Ptolemy, and tangency in the Valencia's sphere

The right triangle defined between the polar line *RW*, and one of the vertices of the major axis of the ellipse, serves to find the height of the polarity director plane. The vertex *U* of the right circular cone which contains the conic section can be located. This height can be measured from the center *R* of the Valencia's sphere.

» Determination of the length of the polar line RW

In the right triangle RV_1W (the cathetus RV_1 is the radius of the Valencia's sphere opposed to the angle β of the polar vertex *W*, and the polar line *RW* is the hypotenuse), *RW* is computed as (figure 10):

$$RW = \frac{RVa}{\sin\beta} \tag{11}$$

By replacing the numerical values, it is obtained: *RW* = 90.9800012.

» Determination of the height of the polarity director plane, with respect to the center of the Valencia's sphere

In the right triangle *RUW* (the cathetus *RU* is the height of the PDP with respect to the center *R* of the Valencia's sphere; and the polar line *RW* is the hypotenuse), *RU* is computed as (figure 10):

$$RU = RW \cdot \cos\alpha \tag{12}$$

By replacing the numerical values, it is obtained: *RU* = 73.60436672.

» The determination of the metric ratios of proportionality in the quadrilateral of Ptolemy

The metric ratios of the quadrilateral of Ptolemy, as well as the convergent secants (defined by the extension of their pairs of sides), will allow the conservation of proportions, as a function of the variation of the angular value α of the secant plane.

The following relations can be written [3] (in fig. 10):

$$V_1R_1 \cdot V_2R_2 + V_1R_2 \cdot V_2R_1 = R_1R_2 \cdot V_1V_2 \tag{13}$$

$$XR_2 \cdot XV_1 = XR_1 \cdot XV_2 \tag{14}$$

$$YR_2 \cdot YV_2 = YR_1 \cdot YV_1 \tag{15}$$

» Determination of the focal radii

The vector rays that start from the poles R_1 and R_2, are perpendicular to the polar circle of tangency that contains the conic section, determining the exact position of the foci of the ellipse. The angle between these vectors with respect to the polar axis of the Valencia sphere, is constant and equal to the angle α of the secant plane.

The orthogonal chords that pass through the focal points F_1 and F_2, define the rectangle R_1SVaR_2, and meet the equality between their pairs of opposite sides. In this way, the focal radius $R_1F_1 = SF_2$, and the focal radius $R_2F_2 = VaF_1$. The straight lines drawn from the corners *S* and *Va*, which pass through one of the vertices of the ellipse, form two equal angles to each other. This angle is measured with respect to the sides of the rectangle R_1S and R_1Va, and is related to the angles (γ = 90º - α), and β. The following relations can be written (in figure 10):

$$\mu = \frac{\gamma - \beta}{2}; \quad \Omega = \frac{\gamma + \beta}{2} \tag{16}$$

Determination of the focal radius R_1F_1:

$$R_1F_1 \cdot F_1Va = V_1F_1 \cdot V_2F_1 \;\; ; \;\; F_1Va = \frac{V_1F_1}{\tan\mu} \;\; ; \;\; R_1F_1 = V_2F_1 \cdot \tan\mu \tag{17}$$

Determination of the focal radius R_2F_2:

$$R_2F_2 \cdot F_2S = V_2F_2 \cdot V_1F_2 \;\; ; \;\; F_2S = \frac{V_2F_2}{\tan\Omega} \;\; ; \;\; R_2F_2 = V_1F_2 \cdot \tan\Omega \tag{18}$$

By replacing the numerical values, are obtained: R_1F_1 = 14.8863443, R_2F_2 = 44.9888425.

— The cylindrical ellipse

A right cylinder could be defined as a right cone, whose vertex is located at infinity [2]. The generatrices are parallel to the axis of revolution, and always in contact with a circular directrix. The conicity angle is equal with 90º.

If the cylindrical surface is intersected by a secant plane with an angle greater than 0º, and smaller than 90º, with respect to the axis of the cylinder, it always determines an ellipse, where each vertex of the major axis of the ellipse is located in a generatrix of the cylindrical surface.

Starting from the same length of the major axis of the ellipse obtained with the right circular cone, and with the same angular value α of the secant plane, we proceed to determine the parameters of the ellipse, with the Valencia's sphere.

If the angle complementary to the conicity β is equal to 0º; it follows that the radius of the sphere is equal to the semi-major axis of the ellipse, as shown in figure 11.

$$RVa = \frac{OV_1}{\cos\beta} \tag{19}$$

By replacing the numerical values, it is obtained: $RVa = OV_1 = 33.80565811$.

This means that the center of the sphere coincides with the center of the ellipse, and the polar axis will be co-linear to the axis of the cylinder. Subsequently, from each pole is drawn the ray perpendicular to the plane containing the ellipse, the angle between the ray and the polar axis being the same angle α of the secant plane.

» Determination of the focal semi-distance of the cylindrical ellipse

In the right triangle R_1OF_1 (the hypotenuse R_1O is the radius of the Valencia's sphere; and the cathetus OF_1 is the focal semi-distance of the ellipse), OF_1 is computed as (fig. 11):

$$OF_1 = RVa \cdot \sin\alpha \tag{20}$$

By replacing the numerical values, it is obtained: $OF_1 = OF_2 = 19.8704672$.

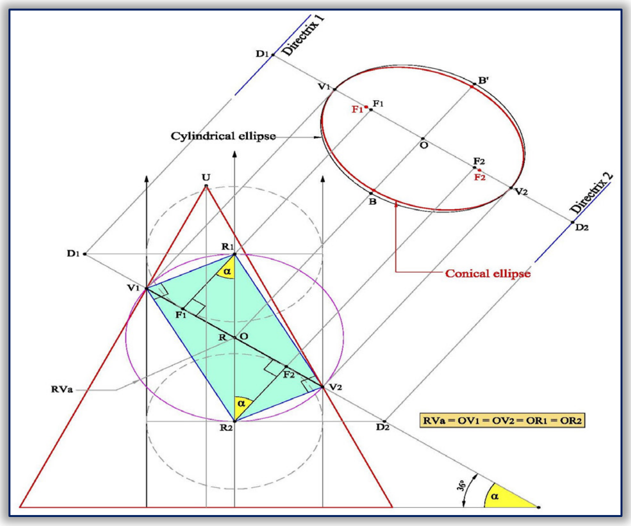

Figure 11. Comparison between a cylindrical and conical ellipse

» Determination of the focal radii of the cylindrical ellipse

"For every cylindrical ellipse, the remarkable quadrilateral of Ptolemy inscribed in the Valencia's sphere, will be always a rectangle[4]*"*, where the property of equality between its pairs of sides is fulfilled. That is to say: $R_1V_1 = R_2V_2$ and $V_1R_2 = V_2R_1$; therefore, the focal radii are equal to each other, and are determined applying the cosine function.

$$R_1F_1 = R_2F_2 = RVa \cdot \cos\alpha \tag{21}$$

By replacing the numerical values, it is obtained: $R_1F_1 = R_2F_2 = 27.3493519$.

» Determination of the minor semi-axis of the cylindrical ellipse

The minor semi-axis of the ellipse is equal to the focal radius, because the Dandelin spheres have the same radius of the cylinder that contains the ellipse.

$$OB = \sqrt{(RVa)^2 - (OF_1)^2} \tag{22}$$

By replacing the numerical values, it is obtained: $OB = 27.3493519 = R_1F_1 = R_2F_2$.

» Determination of the eccentricity and directrices of the cylindrical ellipse

The eccentricity, depending on the focal semi-distance and the semi-major axis of the ellipse:

$$\varepsilon = c/a \tag{23}$$

By replacing the numerical values, it is obtained: $\varepsilon = 0.587785249$.

The distance d with respect to the nearest vertex is computed with relation:

$$d = \frac{a^2 - ac}{c};\ \ d = \frac{OV_1{}^2 - OV_1 \cdot OF_1}{OF_1} \tag{24}$$

By replacing the numerical values, it is obtained: $d = 23.70796294$.

The distance between the directrix and the center of the Valencia's sphere is:

$$OD_1 = \frac{RVa}{\sin\alpha} \tag{25}$$

By replacing the numerical values, it is obtained: $OD_1 = 57.513621$.

4. CONCLUSION

The Valencia's sphere, together with the properties of the remarkable quadrilateral of Ptolemy, represents a very simple and precise geometric method, for the determination of the ellipse's parameters, without to use a Cartesian coordinate system. This is possible if the variables of: the length of the major axis of the ellipse, the conicity angle, and the angle of the secant plane containing the conic are known; as it happens in three-dimensional space.

The parameterization of the ellipse, obtained from the Valencia's sphere, can be compared with the algebraic parameters found by mathematical equations, and thus, to perform a study that unifies the two-dimensional (2D) analysis, with the three-dimensional (3D) aspects involved in the generation of the conics.

In equality conditions, when comparing a conical ellipse with a cylindrical ellipse, it follows that they differ in their eccentricity; therefore, it is necessary to clarify this observation within the mathematical and geometric context; and thus be able to resolve the concern about, if: "The orbits of the planets, describe conical or cylindrical elliptical trajectories? and if "Are the corrections of the trajectories partly due to this justification?"

The concepts used in the Valencia's sphere, can be incorporated into the design of a software or application appropriate for Astronomy and Physics, which allows determining various aspects, such as the parameters of the elliptical trajectories, distances relative to the polarity director plane, perihelion and aphelion, eccentricities, focal radii, polar distances from the geometric center of the ellipse, focal semi-distances, semi-axes, distance to the directrices, etc. The proposed methodology can be implemented in the curricula of a curriculum, as a new pedagogical resource, in the subjects related to the study and analysis of the ellipses, where the other focal-type conics are explored, while retaining, a three-dimensional approach.

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