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SOLITON SOLUTIONS OF NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS OF FRACTIONAL ORDER

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Abstract: Nonlinear mathematical problems and their solutions attain much attentions. In soliton theory an efficient tool to attain various types of soliton solutions is the Exp-function technique. Under study article is devoted to find soliton wave solutions of YTSF equation and Calogero-Bogoyavlenskii-Schiff equation (CBS) of fractional-orders via a reliable mathematical technique. By use of proposed technique we attains soliton wave solution of various types. The regulation of proposed algorithm is demonstrated by consequent numerical results and computational work. It is concluded that under discussion technique is user friendly with minimum computational work, also we can extend for physical problems of different nature.
Keywords: modified Riemann-Liouville derivative, potential-YTSF equation, Calogero-Bogoyavlenskii-Schiff equation (CBS), Exp-function method, fractional calculus

1. INTRODUCTION

In the last few years we have observed an extraordinary progress in soliton theory. Solitons have been studied by various mathematician, physicists and engineers for their applications in physical phenomena's. Firstly soliton waves are observed by an engineer John Scott Russell. Wide ranges of phenomena in mathematics and physics are modeled by differential equations. In nonlinear science it is of great importance and interest to explain physical models and attain analytical solutions. In the recent past large series of chemical, biological, physical singularities are feint by nonlinear partial differential equations. At present the prominent and valuable progress are made in the field of physical sciences. The great achievement is the development of various technique to hunt for solitary wave solutions of differential equations. In nonlinear physical sciences, an essential contribution is of exact solutions because of this we can study physical behaviors and discus more features of the problem which give direction to more applications.

At the disruption between chaos, mathematical physics and probability, factional calculus and differential equations are rapidly increasing branches of research. For accurate clarification of innumerable real-time models of nonlinear occurrence fractional differential equation (NFDEs) of nonlinear structure have accomplished great notice. Because of its recognizable implementation in branch of sciences and engineering it turn out to be a topic of great notice for scientists in workshops and conferences. In large fields such as porous structures dynamical processes in self-similar or solute transport and fluid flow, material viscoelastic theory, economics, bio-sciences, control theory of dynamical systems, geology, diffusive transport akin to diffusion, electromagnetic theory, dynamics of earthquakes, statistics, astrophysics, optics, probability, signal processing and chemical physics, and so on implementations of fractional models [5-8] are beneficially exerted. As a consequence, hypothesis of fractional differential equations has shown fast growth, see [1-8]

In current times, to solve a nonlinear physical problems Wu and He [9] present a well-ordered procedure called Exp-function method. The technique under study has prospective to deal with the complex nonlinearity of the models with the flexibility. It has been used as an effective tool for diversified nonlinear problems arising in mathematical physics. Through the study of publication exhibits that Exp-function method is extremely reliable and has been effective on a huge range of differential equations.

After He et al. Mohyud-Din enlarged the Exp-function method and used this algorithm to find soliton wave solutions of differential equations; Oziz used same technique for Fisher's equation; Yusufoglu for MBBN equations; for non-linear higher-order boundary value physical problems ; Momani for travelling wave solutions of KdV equation of fractional order; Zhu for discrete m KdV lattice and the Hybrid-Lattice system; Kudryashov for soliton solutions of the generalized evolution equations arising in wave dynamics ;Wu et al. for the expansion of compaction-like solitary and periodic solutions; Zhang for high-dimensional nonlinear differential equations. It is to be noticed that after applying under study technique i.e. Exp-function method to any ordinary nonlinear differential equation Ebaid proved that $s = r$ and $l = j$ are the only relations of the variables that can be acquired [10-25]. This article is keen to the solution like solutions of nonlinear Calogero--Bogoyavlenskii--Schiff equation and for nonlinear potential-YTSF equation of fractional orders by applying a novel technique, the applications of under study nonlinear equation are very vast in different areas of physical sciences and engineering. Additionally, such type of equation found in different physical phenomenon related to fluid mechanic, astrophysics, solid state physics, and chemical kinematics, ion acoustic waves in plasma, control and optimization theory, nonlinear optics etc. Exp-function method using Jumarie's subsequent application [26-33] and its derivative approach.

Schiff and Bogoyavlenskii explore the Calogero--Bogoyavlenskii--Schiff equation (CBS) by so many different ways. Bogoyavlenskii used the modified Lax formulation and Schiff attained this equation by minimizing the self-dual Yang-Mills equation [30-31]. The established composition can be expended to other physical models appearing in engineering, applied sciences and mathematical physics. It is noticed that this composition is very consistent, fully compatible and extremely well-organized for nonlinear differential equations of fractional-order.

2. JUMARIE'S FRACTIONAL DERIVATIVE

We define modified Riemann-Liouville derivative given as

$$D_t^{\beta}h(x)=\begin{cases}\dfrac{1}{\Gamma(-\beta)}\displaystyle\int_0^x (x-t)^{-\beta-1}(h(t)-h(0))dt,\beta\le 0\\ \dfrac{1}{\Gamma(-\beta)}\dfrac{d}{dx}\displaystyle\int_0^x (x-t)^{-\beta-1}(h(t)-h(0))dt,0\le\beta\le 1\\ \overline{}\\ [h^{\beta-q}(x)^q]^q,q\le\beta\le q+1,q\ge 1\end{cases} \tag{1}$$

Here $h:R\to R,x\to h(x)$ is a function which is continuous (not differentiable). There are few results [26-29], which are very important and useful

$$D_x^{\beta}l=0,\beta\ge 0, \tag{2}$$

$$D_x^{\beta}[lh(x)]=lD_x^{\beta}\beta\ge 0, \tag{3}$$

$$D_x^{\beta}x^{\zeta}=\frac{\Gamma(1+\beta)}{\Gamma(1+\beta-\beta)}x^{\zeta-\beta},\beta\ge\zeta\ge 0 \tag{4}$$

$$D_x^{\beta}[h(x)l(x)]=[j_x^{\beta}h(x)]l(x)+h(x)[D_x^{\beta}l(x)] \tag{5}$$

$$D_x^{\beta}h(x(t))=h_x^{'}x^{\beta}(t) \tag{6}$$

3. EXP-FUNCTION TECHNIQUE

We suppose the nonlinear FPDE of the form

$$Q(\eta,\eta_x,\eta_{xx},\eta_{xxx},...,D_t^{\beta},D_x^{\beta}\eta,D_{xx}^{\beta}\eta,...)=0,0\le\beta\le 1 \tag{7}$$

where $D_t^{\beta},D_x^{\beta}\eta,D_{xx}^{\beta}\eta$ are the modified Riemann-Liouville derivative of η w.r.t t,x,xx respectively.

Invoking the transformation

$$\xi=cx+py+qz+\frac{\omega t^{\beta}}{\Gamma(1+\beta)}+\xi_0 \tag{8}$$

Here c,p,q,ω,ξ_0 are all constants with $c,\omega\ne 0$

We can write equation (7) again in the form of following nonlinear ODE

$$S(\eta,\eta',\eta'',\eta''',\eta^{iv}\dots)=0 \tag{9}$$

Where prime signify the derivative of η with respect to . In proportion to Exp-function method, we suppose that the wave solution can be written in the form given below

$$\eta(\xi)=\frac{\sum_{q-l}^{j}u_q\exp[q\xi]}{\sum_{p-r}^{s}v_p\exp[p\xi]} \tag{10}$$

Where r,s,l and j are positive integers which we have to find, u_q and v_p are unknown constants. We can write equation (10) again in the following equivalent form

$$\eta(\xi)=\frac{u_l\exp(l\xi)+...+u_{-j}\exp(-j\xi)}{v_r\exp(r\xi)+...+u_{-s}\exp(-s\xi)} \tag{11}$$

This equivalent transformation plays a fundamental and important part to solve the problem for analytical solution. To find the value of l and r by using [25], we have

$$r=l,s=j \tag{12}$$

4. NUMERICAL APPLICATIONS

We apply Exp-function technique to attain soliton wave solution of potential-YTSF equation and Calogero--Bogoyavlenskii--Schiff equation of fractional-orders. The obtained results are very efficient and encouraging.

— Suppose the following Calogero--Bogoyavlenskii--Schiff equation of fractional order

$$D_x^{\beta}\eta_x + \eta_{xxxz} + 4\eta_x\eta_{xz} + 2\eta_{xx}\eta_z = 0, 0 \le \beta \le 1 \tag{13}$$

Using (8) equation (13) can be converted into an ODE of the form

$$\omega c\eta''' + c^3 p\eta^{(iv)} + 4c^2 p\eta''' + 2c^2 p\eta''' = 0 \tag{14}$$

Where prime signify the derivative of η w.r.t ξ. Equation (11) is the expressed solution of the equation (13) .To find the value of r and s, by using [25]

$$r = l, s = j \tag{15}$$

≡ Case I. We can frequently select the parameters l and j, For directness, we set $r = l = 1$ and $s = j = 1$ equation (11) becomes

$$\eta(\xi) = \frac{u_1 \exp[\xi] + \ldots + u_{-1}\exp[-\xi]}{v_1 \exp[\xi] + \ldots + u_{-1}\exp[-\xi]} \tag{16}$$

Using equation (16) into equation (14), we have

$$\frac{1}{A}\begin{bmatrix} l_4\exp(4\xi) = l_3\exp(3\xi) + l_2\exp(2\xi) + l_1\exp(\xi) + l_0 + l_{-1}\exp(-\xi) + l_{-2}\exp(-2\xi) \\ + l_{-3}\exp(-3\xi) + l_{-4}\exp(-4\xi) \end{bmatrix} = 0 \tag{17}$$

where $A = (v_1\exp(\xi) + v_0 + v_{-1}\exp(-\xi))^4$, l_i are constants whose values are attained by Maple 16. By equalizing the coefficients of $\exp(q\xi)$ to zero, we get

$$\{l_{-4} = 0, l_{-3} = 0, l_{-2} = 0, l_{-1} = 0, l_0 = 0, l_1 = 0, l_2 = 0, l_3 = 0, l_4 = 0\} \tag{18}$$

Which is solution of (18) the given equation (13) is satisfied by five solution sets.

» 1st Solution set:

$$\left\{\omega = \omega, u_{-1} = \frac{u_0 v_{-1}}{v_0}, u_0 = u_0, u_1 = 0, v_0 = v_0, v_{-1} = v_{-1}, v_1 = 0\right\}$$

Using above the solution $\eta(x,t)$ of equation (13) is

$$\eta(x,t) = \frac{\frac{-v_{-1}(4cv_1 - u_1)}{v_1} e^{-\frac{xc+pz+4pH^2t^\beta}{\Gamma(1+\beta)}} + u_1 e^{\frac{xc+pz+4pH^2t^\beta}{\Gamma(1+\beta)}}}{v_{-1}e^{-\frac{xc+pz+4pH^2t^\beta}{\Gamma(1+\beta)}} + v_1 e^{\frac{xc+pz+4pH^2t^\beta}{\Gamma(1+\beta)}}}$$

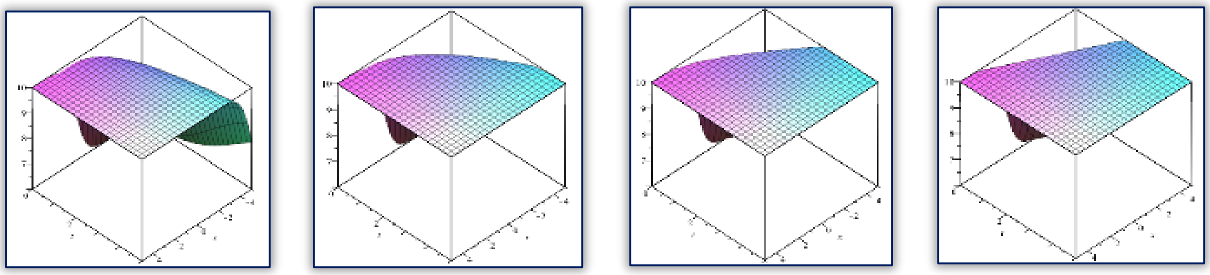

Figure 1. 1st Solution set: β = .25; β = .50; β = .75; β = 1

» 2nd Solution set:

$$\left\{\omega = 4pc^2, u_{-1} = \frac{-v_{-1}(4cv_1 - u_1)}{v_1}, u_0 = 0, u_1 = u_1, v_{-1} = v_{-1}, v_0 = 0, v_1 = v_1\right\}$$

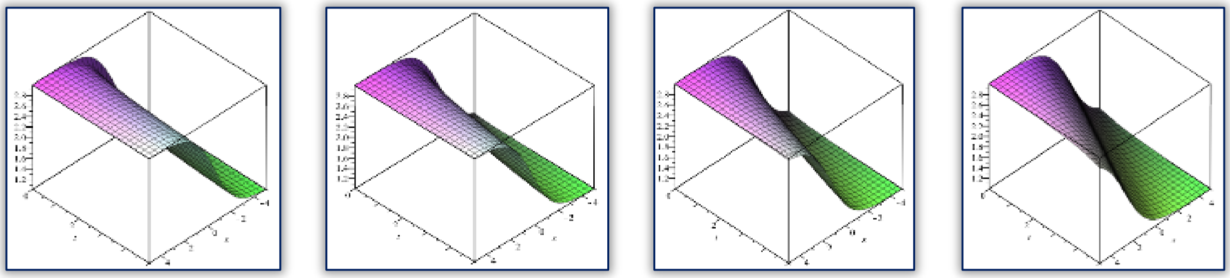

Figure 2. 2nd Solution set: β = .25; β = .50; β = .75; β = 1

Therefore we attain following $\eta(x,t)$ given here: $\eta(x,t)=\dfrac{\frac{-v_{-1}(2cv_0-u_0)}{v_0}e^{-\frac{xc+pz+pH^2t^\beta}{\Gamma(1+\beta)}}+u_0}{v_{-1}e^{-\frac{xc+pz+pH^2t^\beta}{\Gamma(1+\beta)}}+v_0}$

» 3rd Solution set:

$$\left\{\begin{array}{c}\omega=pH^2,u_0=u_0,u_1=u_1,v_0=v_0,v_1=v_1\\ v_{-1}=-\frac{1}{4}\frac{2Hu_0v_1^2v_0-2Hu_1v_0^2v_1+H_1^2v_0^2+u_0^2v_1^2-2u_1v_0u_0v_1}{c^2v_1^3}\\ u_{-1}=\frac{1}{4}\frac{1}{H^2v_1^4}(-6u_1v_0u_0v_1^2H+4Hv_1v_0^2u_1^2-u_1^3v_0^2-u_1u_0^2v_1^2+2u_1^2v_0u_0v_1+4H^2u_0v_1^3v_0-4H^2v_1^2v_0^2u_1+2u_0^2v_1^3H\end{array}\right\}$$

Therefore we get the following generalized unique solution $\eta(x,t)$ of equation (13)

$$\eta(x,t)=\frac{u_{-1}e^{-\frac{xc+pz+4pH^2t^\beta}{\Gamma(1+\beta)}}+u_0+u_1e^{\frac{xc+pz+4pH^2t^\beta}{\Gamma(1+\beta)}}}{v_{-1}e^{-\frac{xc+pz+4pH^2t^\beta}{\Gamma(1+\beta)}}+v_0+v_1e^{\frac{xc+pz+4pH^2t^\beta}{\Gamma(1+\beta)}}}$$

where $v_{-1}=-\frac{1}{4}\frac{2Hu_0v_1^2v_0-2Hu_1v_0^2v_1+u_1^2v_0^2+u_0^2v_1^2-2u_1v_0u_0v_1}{H^2v_1^3}$

$$u_{-1}=\frac{1}{4}\frac{1}{c^2v_1^4}(-6u_1v_0u_0v_1^2H+4Hv_1v_0^2u_1^2-u_1^3v_0^2-u_1u_0^2v_1^2+2u_1^2v_0u_0v_1+4H^2u_0v_1^3v_0-4c^2v_1^2v_0^2u_1+2u_0^2v_1^3H$$

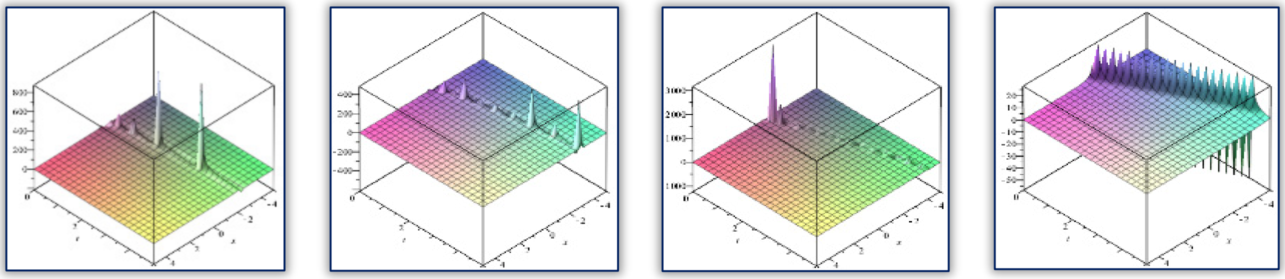

Figure 3. 3rd Solution set: β = .25; β = .50; β = .75; β = 1

≡ Case II. If $r=l=2$ and $s=j=1$ then equation (13) takes the form

$$\eta(\xi)=\frac{u_2\exp[2\xi]+u_1\exp[\xi]+u_0+u_{-1}\exp[-\xi]}{v_2\exp[2\xi]+v_1\exp[\xi]+v_0+u_{-1}\exp[-\xi]} \quad (19)$$

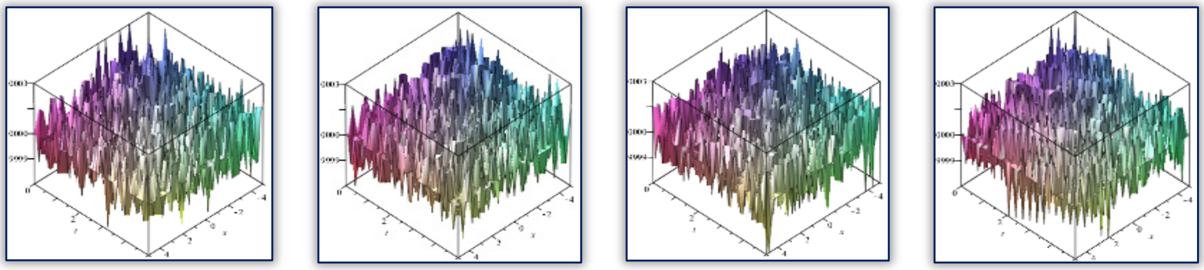

Figure 3. Case II: β = .25; β = .50; β = .75; β = 1

Proceeding as recent, we get

$$\left\{\omega=\omega,u_{-1}=0,u_2=\frac{u_1v_2}{v_1},u_0=\frac{u_1v_0}{v_1},u_1=u_1,v_0=v_0,v_{-1}=0,v_1=v_1,v_2=v_2\right\}$$

we attain the soliton solution of equation (13): $\eta(x,t)=\dfrac{\frac{v_0u_2}{v_2}+\frac{u_2v_1}{v_2}e^{\frac{xc+pz+\omega t^\beta}{\Gamma(1+\beta)}}+u_2e^{\frac{2(xc+pz+\omega t^\beta)}{\Gamma(1+\beta)}}}{v_0+v_1e^{\frac{xc+pz+\omega t^\beta}{\Gamma(1+\beta)}}+v_2e^{\frac{2(xc+pz+\omega t^\beta)}{\Gamma(1+\beta)}}}$

— Suppose the following (3 + 1)-dimensional potential-YTSF equation

$$-4D_t^\beta\eta_x+\eta_{xxxz}+4\eta_x\eta_{xz}+2\eta_{xx}\eta_z+3\eta_{yy}=0, 0\le\beta\le1 \quad (20)$$

By using equation (8) equation (20) can be converted to an ODE given below

$$-4\omega c\eta''+c^3q\eta^{(iv)}+4c^2q\eta'''+2c^2q\eta'''+3p^2\eta''=0 \quad (21)$$

Where prime signify the derivative of η with respect to ξ. Equation (11) is the expressed solution of the equation (24) .To find the value of l and r, by using [25]

$$r = l, s = j \tag{22}$$

≡ **Case I.** The values of l and j are chosen arbitrary. For our ease, we set $r = l = 1$ and $s = j = 1$ equation (11) reduces to

$$\eta(\xi) = \frac{u_1 \exp[\xi] + u_0 + u_{-1} \exp[-\xi]}{v_1 \exp[\xi] + v_0 + u_{-1} \exp[-\xi]} \tag{23}$$

Inserting equation (27) into (25),

$$\frac{1}{A}\begin{bmatrix} l_4 \exp(4\xi) = l_3 \exp(3\xi) + l_2 \exp(2\xi) + l_1 \exp(\xi) + l_0 + l_{-1} \exp(-\xi) + l_{-2} \exp(-2\xi) \\ + l_{-3} \exp(-3\xi) + l_{-4} \exp(-4\xi) \end{bmatrix} = 0 \tag{24}$$

where $A = (v_1 \exp(\xi) + v_0 + v_{-1} \exp(-\xi))^4$, l_i are constants that we get by Maple 16. Equalizing the coefficients of $\exp(q\xi)$ to zero, we get

$$\{l_{-4} = 0, l_{-3} = 0, l_{-2} = 0, l_{-1} = 0, l_0 = 0, l_1 = 0, l_2 = 0, l_3 = 0, l_4 = 0\} \tag{25}$$

Solution of (25) we have five solution set satisfy the given equation

» **1st Solution set:**

$$\left\{\omega = \omega, u_{-1} = u_{-1}, u_0 = 0, u_1 = u_1, v_0 = 0, v_{-1} = v_{-1}, v_1 = \frac{u_1 v_{-1}}{u_{-1}}\right\}$$

The solution $\eta(x,t)$ of equation (20) is given here: $\eta(x,t) = \dfrac{u_0 + u_1 e^{\frac{xc+py+qz+\frac{1}{4}\frac{H^3 q+3p^2}{H} t^\beta}{\Gamma(1+\beta)}}}{v_0 + \frac{u_1 v_0}{2v_0 c+u_0} e^{\frac{xc+py+qz+\frac{1}{4}\frac{H^3 q+3p^2}{H} t^\beta}{\Gamma(1+\beta)}}}$

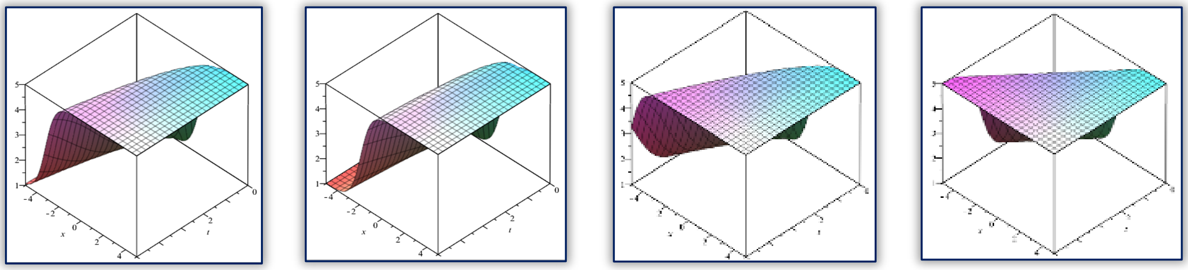

Figure 4. 1st Solution set: β = .25; β = .50; β = .75; β = 1

» **2nd Solution set:**

$$\left\{\begin{array}{l} \omega = \omega, u_{-1} = \dfrac{u_0 v_{-1}}{v_0} \\ u_0 = u_0, u_1 = -\dfrac{1}{11} \dfrac{\frac{(-7v_1 v_{-1} v_0 + v_0^3) u_0 v_{-1}}{v_0} + (-4v_1 v_{-1}^2 - v_0^2 v_{-1}) v_1 u_0}{v_0 v_{-1}^2}, v_{-1} = v_{-1}, v_0 = v_0, v_1 = v_1 \end{array}\right\}$$

Generalized solitary solution $\eta(x,t)$ is $\eta(x,t) = \dfrac{u_{-1} e^{-\frac{xc+py+qz+\omega t^\beta}{\Gamma(1+\beta)}} + u_0 + u_1 e^{\frac{xc+py+qz+\omega t^\beta}{\Gamma(1+\beta)}}}{v_{-1} e^{-\frac{xc+py+qz+\omega t^\beta}{\Gamma(1+\beta)}} + v_0 + v_1 e^{\frac{xc+py+qz+\omega t^\beta}{\Gamma(1+\beta)}}}$

where $u_1 = -\dfrac{1}{11} \dfrac{\frac{(-7v_1 v_{-1} v_0 + v_0^3) u_0 v_{-1}}{v_0} + (-4v_1 v_{-1}^2 - v_0^2 v_{-1}) v_1 u_0}{v_0 v_{-1}^2}, u_{-1} = \dfrac{u_0 v_{-1}}{v_0}$

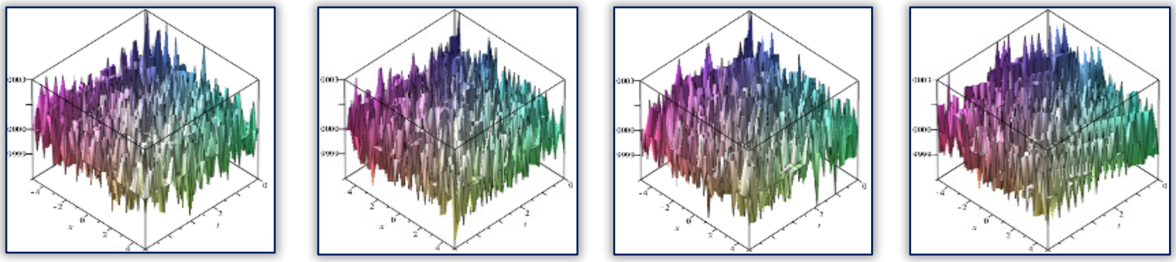

Figure 5. 2nd Solution set: β = .25; β = .50; β = .75; β = 1

» 3rd Solution set:

$$\left\{\begin{array}{l}\omega=\dfrac{1}{4}\dfrac{H^3q+3p^2}{H}, u_{-1}=u_{-1}\\ u_0=-\dfrac{2\sqrt{-(2v_{-1}c+u_{-1})u_1cv_{-1}}}{2v_{-1}c+u_{-1}} u_1=u_1, v_{-1}=v_{-1}, v_0=0, v_1=\dfrac{v_{-1}u_1}{2v_{-1}c+u_{-1}}\end{array}\right\}$$

We attain the following soliton solution $\eta(x,t)$: $\eta(x,t)=\dfrac{u_{-1}e^{-\frac{xc+py+qz+\omega t^{\beta}}{\Gamma(1+\beta)}}-u_0+u_1e^{\frac{xc+py+qz+\omega t^{\beta}}{\Gamma(1+\beta)}}}{v_{-1}e^{-\frac{xc+py+qz+\omega t^{\beta}}{\Gamma(1+\beta)}}+v_1e^{\frac{2(xc+py+qz+\omega t^{\beta})}{\Gamma(1+\beta)}}}$

where $\omega=\dfrac{1}{4}\dfrac{H^3q+3p^2}{H}, u_0=-\dfrac{2\sqrt{-(2v_{-1}c+u_{-1})u_1cv_{-1}}}{2v_{-1}c+u_{-1}}, v_1=\dfrac{v_{-1}u_1}{2v_{-1}c+u_{-1}}$

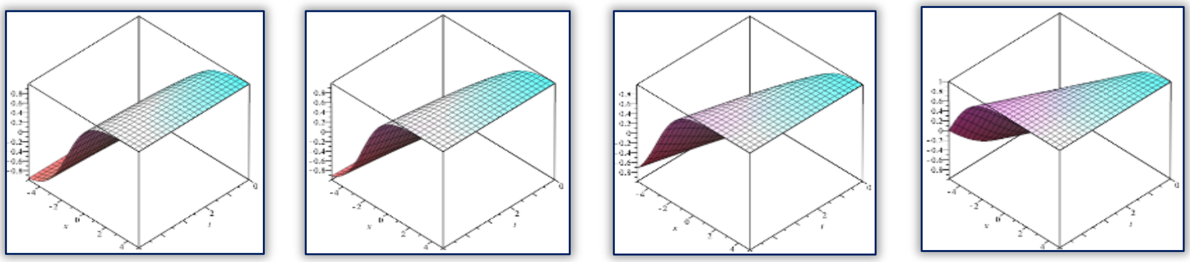

Figure 6. 3rd Solution set: β = .25; β = .50; β = .75; β = 1

» 4th Solution set:

$$\left\{\begin{array}{l}\omega=\dfrac{1}{4}\dfrac{H^3q+3p^2}{H}, u_{-1}=u_{-1}\\ u_0=-\dfrac{2\sqrt{-(2v_{-1}c+u_{-1})u_1cv_{-1}}}{2v_{-1}c+u_{-1}} u_1=u_1, v_{-1}=v_{-1}, v_0=0, v_1=\dfrac{v_{-1}u_1}{2v_{-1}c+u_{-1}}\end{array}\right\}$$

Therefore, we get the following generalized solution $\eta(x,t)$ of equation (20)

$$\eta(x,t)=\frac{u_{-1}e^{-\frac{xc+py+qz+\omega t^{\beta}}{\Gamma(1+\beta)}}+u_0+u_1e^{\frac{xc+py+qz+\omega t^{\beta}}{\Gamma(1+\beta)}}}{v_{-1}e^{-\frac{xc+py+qz+\omega t^{\beta}}{\Gamma(1+\beta)}}+v_1e^{\frac{2(xc+py+qz+\omega t^{\beta})}{\Gamma(1+\beta)}}}$$

where $\omega=\dfrac{1}{4}\dfrac{H^3q+3p^2}{H}, u_0=-\dfrac{2\sqrt{-(2v_{-1}c+u_{-1})u_1cv_{-1}}}{2v_{-1}c+u_{-1}}, v_1=\dfrac{v_{-1}u_1}{2v_{-1}c+u_{-1}}$

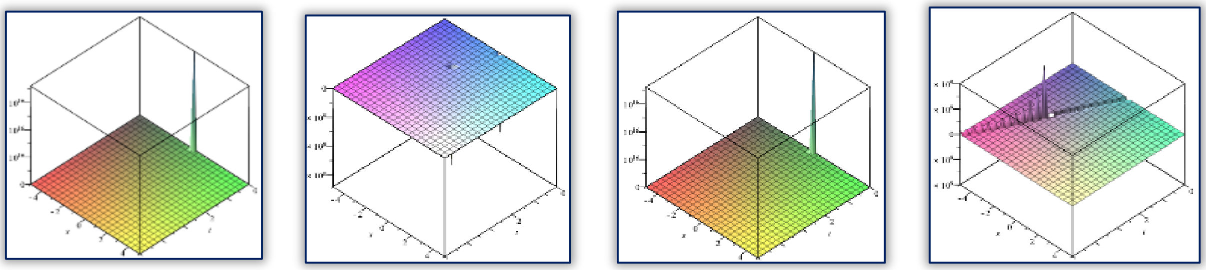

Figure 7. 4th Solution set: β = .25; β = .50; β = .75; β = 1

Case II. By setting $r=l=2$ and $s=j=1$ equation (20) reduces to

$$\eta(\xi)=\frac{u_2\exp[2\xi]+u_1\exp[\xi]+u_0+u_{-1}\exp[-\xi]}{v_2\exp[2\xi]+v_1\exp[\xi]+v_0+u_{-1}\exp[-\xi]} \tag{26}$$

Proceeding as recent, we get

$$\left\{\omega=\omega, u_{-1}=u_{-1}, u_0=0, u_1=u_1, u_2=u_2, v_{-1}=\frac{u_{-1}v_1}{v_1}, v_0=0, v_1=v_1, v_2=\frac{v_1u_2}{u_1}\right\}$$

Hence we obtain the soliton wave solution of equation (20)

$$\eta(x,t)=\frac{u_{-1}e^{-\frac{xc+py+qz+\omega t^{\beta}}{\Gamma(1+\beta)}}+u_1e^{\frac{xc+py+qz+\omega t^{\beta}}{\Gamma(1+\beta)}}+u_2e^{\frac{2(xc+py+qz+\omega t^{\beta})}{\Gamma(1+\beta)}}}{\frac{u_{-1}v_1}{u_1}e^{-\frac{xc+py+qz+\omega t^{\beta}}{\Gamma(1+\beta)}}+v_1e^{\frac{xc+py+qz+\omega t^{\beta}}{\Gamma(1+\beta)}}+\frac{v_1u_2}{u_1}e^{\frac{2(xc+py+qz+\omega t^{\beta})}{\Gamma(1+\beta)}}}$$

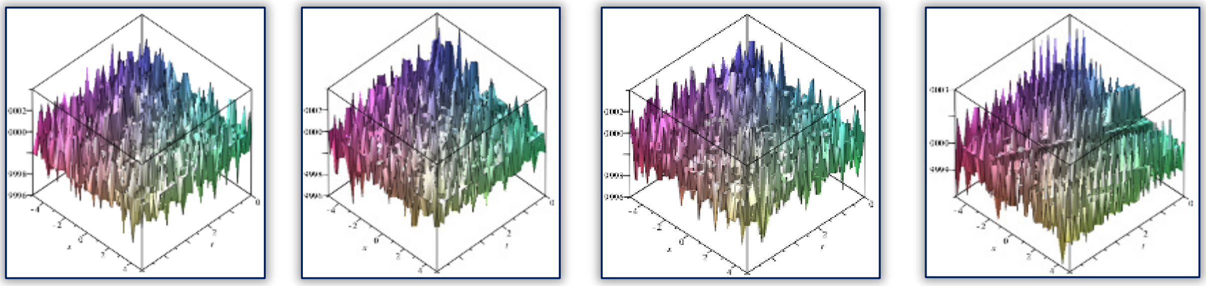

Figure 8. Case II: β = .25; β = .50; β = .75; β = 1

5. RESULTS AND DISCUSSION

From the above figures we note that soliton is a wave which preserves its shape after it collides with another wave of the same kind. By solving Calogero--Bogoyavlenskii--Schiff equation and nonlinear potential-YTSF equation of fractional order, we attain desired soliton wave solutions. The solitary wave moves toward right if the velocity is positive or left directions if the velocity is negative and the amplitudes and velocities are controlled by various parameters. Solitary waves show more complicated behaviors which are controlled by various parameters. Figures signify graphical representation for different values of parameters. In both cases, for various values of parameters l, r, j and s we get the identical soliton solutions which clearly understand that final solution does not effectively based upon these parameters. So we can choose arbitrary values of such parameters. Since the solutions depend on arbitrary functions, we choose different parameters as input to our simulations.

6. CONCLUSIONS

This article is devoted to attain, test and analyze the novel soliton wave solutions and physical properties of nonlinear partial differential equation. For this, fractional order potential-YTSF equation and the nonlinear Calogero—Bogoyavlenskii-Schiff (CBS) equation is considered and we apply Exp function method. We attain desired soliton solutions of various types for different values of parameters. It is guaranteed the accuracy of the attain results by backward substitution into the original equation with Maple 16. The scheming procedure of this method is simplest, straight and productive. We observed that the under study technique is more reliable and have minimum computational task, so widely applicable. In precise we can say this method is quite competent and much operative for evaluating exact solution of NLEEs. The validity of given algorithm is totally hold up with the help of the computational work, the graphical representations and successive results. Results obtained by this method are very encouraging and reliable for solving any other type of NLEEs. The graphical representations clearly indicate the solitary solutions. It is noticed that Exp-function method is very useful for finding solutions of a huge class of nonlinear models and very suitable to apply.

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