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ON THE CUT-OFF FREQUENCIES FOR HEAT CONDUCTING ISOTROPIC PLATES IN GENERALIZED THERMOELASTICITY

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Abstract: Asymptotic analysis of the generalized thermoelasticity equations is carried out for zero surface traction and heat conducting isotropic plates in the vicinity of the cut-off frequencies. Thickness of shear resonance frequencies are also studied in the context of generalized thermoelasticity. The exact dispersion relations are also derived giving frequency as a function of wave number. On taking the thermal relaxation time zero, the obtained results in the analysis reduce to the conventional coupled theory of thermoelasticity. Similarly when the coupling constant is taken identically equal to zero, the strain and thermal fields are uncoupled from each other. In this case, the results can be obtained from the uncoupled theory of thermoelasticity.
Keywords: asymptotic, generalized thermoelasticity, dispersion, thermal relaxation time, frequencies

1. INTRODUCTION

The use of fiber-reinforced composites is prevalent in modern structures, especially those for which a strength-to-weight ratio is of primary concern. For such materials a preferred direction, usually termed the fiber direction, exists and a transversely isotropic model is most commonly employed. For many applications fiber-reinforced materials are formed into bonded layers, each with a specific fiber orientation relative to some fixed reference direction. The practical importance of such structure has resulted in a high number of publications aimed at elucidating the mechanical and dynamic properties of layered media, many including layers composed of fiber-reinforced material. In dynamic problems concerning fiber-reinforced media it is common to use a continuum model, whereby the fibers are assumed to be an inherent material property, rather than some form of inclusion. Wave propagations in unbounded homogeneous layered media have been well known, and many methods have been proposed. The exact dispersion relations can be found in many good books as [1, 2].

The heat conduction equations for uncoupled and coupled theories of thermo-elasticity are of the diffusion type and predict an infinite speed of propagation of the heat wave which is physically unacceptable. To get rid of this absurdity of the classical approach, theories of generalized thermoelasticity were developed. Presently there are various gene-ralized approaches but the theories proposed by Lord and Shulman [8] and Green and Lindsay [9] are called LS and GL theories respectively are most popular. These theories have been developed by introducing one or two relaxation times in the thermoelastic processes; intending to eliminate the paradox of an infinite speed for the propagation of thermal signals.

The LS model is based on a modified Fourier's law, but the GL model even allows second sound without violating the classical Fourier's law. These models of generalized thermoelasticity are structurally dissimilar, and one cannot be obtained as a particular case of the other, yet they ensure finite speeds of propagation for thermal wave. Wave types occurring in bounded layered media are very complicated, and in thermoelasticity, the problem becomes even more complicated, because in thermo-elasticity, solutions to both the heat conduction and thermoelasticity problems for all the layers are required. Thermal and mechanical boundary and interface conditions are also to satisfy by these solutions. As a result, conventional procedure for thermoelastic analysis of multilayered medium resulted in having to solve system of two simultaneous equations for a large number of unknown constants [3-7].

The classical theory of dynamic thermo-elasticity that takes into account the coupling effects between temperature and strain fields involves the infinite thermal wave speed. The theories of generalized thermoelasticity have been developed in an attempt to eliminate this paradox of infinite velocity of thermal pro-pagation. At present, there are two theories on the generalized thermoelasticity: the first is by Lord and Shulman [8], and the second by Green and Lindsay [9]. Verma and Hasebe [10] studied the propagation of thermoelastic vibrations in plates using [8]. Verma et al. have studied wave propagation in anisotropic media in the context of generalized thermoelasticity with different hypotheses [11-15].

In this paper, an asymptotic analysis of the equations of thermoelasticity, for a heat conducting isotropic plate and zero surface traction is carried out in the vicinity of the cut-off frequencies, family of thickness shear resonance frequencies are studied in the context of generalized thermoelasticity. The exact dispersion relation is also derived giving frequency as a function of wave number. We begin with brief derivation of the exact dispersion relation associated with small amplitude vibrations of isotropic thermoelastic layer. The long-wave high frequency approximations of the dispersion relations the associated coupled thermoelastic vibrations are derived. Comparison of the numerical and asymptotic solutions of the dispersion relation for cut-off frequencies is also made.

2. FORMULATION

The generalized coupled field governing dynamic thermoelastic processes for homogeneous isotropic materials and in the absence of body forces and heat source for Lord and Schulman [8], theory can be written as:

$$\mu\nabla\mathbf{u}+(\lambda+\mu)\nabla\nabla\cdot\mathbf{u}-\gamma\nabla T=\rho\ddot{\mathbf{u}} \tag{1}$$

$$K\nabla^2 T+\rho C(\dot{T}+\tau_0\ddot{T})=\gamma T_0[(\nabla\dot{\mathbf{u}}+\tau_0\nabla\ddot{\mathbf{u}}) \tag{2}$$

where

$$\gamma=(3\lambda+2\mu)\alpha_t \tag{3}$$

λ, μ are Lamé's parameters. ρ is the density of the medium. C_e and τ_0 are the specific heat at constant strain and thermal relaxation time, respectively. K and α_t are the coefficient thermal conductivity and linear thermal expansion, respectively an overdot denotes the partial derivative with respect to the time variable t.

We define the following dimensionless quantities**:**

$$x_1^*=\frac{v_1}{k_1}x_1,\, x_3^*=\frac{v_1}{k_1}x_3,\, t^*=\frac{v_1^2}{k_1}t,\ u_1^*=\frac{v_1^3\rho}{k_1\beta T_0}u_1,\, u_3^*=\frac{v_1^3\rho}{k_1\beta T_0}u_3,$$

$$\varepsilon_1=\frac{\beta^2 T_0}{\rho C_e v_1^2},\ T^*=\frac{T}{T_0},\ \tau_0^*=\frac{v_1^2}{k_1}\tau_0,\ c_2=\frac{\mu}{2(\lambda+2\mu)},\ c_3=1-c_2 \tag{4}$$

Here

$$v_1=\left(\frac{\lambda+2\mu}{\rho}\right)^{1/2} \tag{5}$$

v_1 is the velocity of compressional waves and $k_1 = K/\rho C_e$ is the thermal diffusivity in the x-direction. Moreover ε_1 is the thermoelastic coupling constant, τ_0^* is the thermal relaxation constant. Introducing the above quantities in (4) and (5) in Eqs. (1) and (2), after suppressing the $*$, can be written as

$$c_2\nabla^2\mathbf{u}+c_3\nabla div\mathbf{u}-\nabla T=\ddot{\mathbf{u}}, \tag{6}$$

$$\nabla^2 T+(\dot{T}+\tau_0\ddot{T})=\varepsilon_1(\nabla\dot{\mathbf{u}}+\tau_0\nabla\ddot{\mathbf{u}}) \tag{7}$$

The stress-displacement, temperature and temperature gradient relevant to our problem in the plate are:

$$\tau_{33}=\left[(1-2c_3^2)\frac{\partial u_1}{\partial x_1}+\frac{\partial u_3}{\partial x_3}-T\right]\gamma T_0, \tag{8}$$

$$\tau_{13}=\gamma T_0 c_3^2\left(\frac{\partial u_1}{\partial x_3}+\frac{\partial u_3}{\partial x_1}\right), \tag{9}$$

$$\tau_{23}=\gamma T_0 c_3^2\left(\frac{\partial u_2}{\partial x_3}+\frac{\partial u_3}{\partial x_2}\right) \tag{10}$$

$$T_{,x_3}=\frac{\partial T}{\partial x_3} \tag{11}$$

3. ANALYSIS

For a plane harmonic wave travelling in the x_1-direction, the solutions u_1, u_2 **and** u_3, and T of Eqs. (6) and (7) take the form:

$$(u_j,T)=(U_j,U_4)e^{k\alpha x_3}e^{i[k\cos(\theta)+k\sin(\theta)-\omega t]},\qquad j=1,\ 2,\ 3.\qquad i=\sqrt{-1}, \tag{12}$$

where k is the wave number and ω is the frequency of a wave travelling with phase velocity $v=\omega/k$ in a direction within the plane of the plate at an angle θ with the x_1-axis and α is to be determined, in terms of k, θ, ω and material constants. From the equation of motion, substituting Eqs. (12) into the equations of motion and heat conduction, Eqs. (6) and (7) yields:

$$T_{mn}(\alpha)U_n=0,\quad m, n=1,2,3,4 \tag{13}$$

where

$$T_{11}=\tilde{c}_3^2(\alpha^2-\sin(\theta)^2)-\cos(\theta)^2+\tilde{v}^2$$

$$T_{12}=-(\tilde{c}_3^2+\tilde{c}_4^2)\sin(\theta)\cos(\theta),\ T_{21}=T_{12},\ T_{31}=T_{13}$$

$$T_{13}=i(\tilde{c}_3^2+\tilde{c}_4^2)\alpha\cos(\theta),\ T_{14}=i\cos(\theta),\ T_{41}=T_{14}$$

$$T_{22}=\tilde{c}_3^2(\alpha^2-\cos(\theta)^2)-\sin(\theta)^2+\tilde{v}^2$$

$$T_{23}=i(1-\tilde{c}_3^2)\alpha\sin(\theta),\ T_{24}=i\sin(\theta)$$

$$T_{33}=\alpha^2-\tilde{c}_3^2+\tilde{v}^2,\ T_{24}=\alpha$$
$$T_{32}=T_{23},T_{42}=T_{24},T_{43}=T_{24}$$
$$T_{44}=\frac{K(\alpha^2-1)-Ce\tau\tilde{v}^2}{\varepsilon Ce\tau\tilde{v}^2} \tag{14}$$

where

$$c_3^2=\mu/(\lambda+2\mu)\rho,\ c_4^2=\lambda/(\lambda+2\mu)\rho, \tag{15}$$
$$\tilde{v}=\omega/(\lambda+2\mu)k \tag{16}$$

The existence of nontrivial solutions for U_1,U_2,U_3 and U_4 requires the vanishing of the determinant Eq. (13), and yields the following polynomial equation:

$$\left[(\lambda+2\mu)\tilde{v}^2+\mu(\alpha^2-1\right]^2\left\{\left(\alpha^2-1+\tilde{v}^2\right)K(\alpha^2-1)-Ce\tau\tilde{v}^2)+(1-\alpha^2)\varepsilon Ce\tau\tilde{v}^2\right\}=0 \tag{17}$$

4. DISPERSION RELATION

If the three distinct roots of the Eq. (17) are denoted by α_1^2, α_2^2 and α_3^2, the solutions u_1,u_2,u_3 and T are obtainable as linear combinations of the linearly independent solutions, thus:

$$U_1=\sum_{m=1}^{3}\left(U_1^{(2m-1)}E_m^+ +U_1^{(2m)}E_m^-\right),$$
$$U_3=\sum_{m=1}^{3}\left(U_1^{(2m-1)}E_m^+ +U_1^{(2m)}E_m^-\right)$$
$$U_4=\sum_{m=1}^{3}\left(U_1^{(2m-1)}E_m^+ +U_1^{(2m)}E_m^-\right) \tag{18}$$

where $E_m^+=\exp(k\alpha_m x_3)$, $E_m^-=\exp(-k\alpha_m x_3)$

(m =1,2,3) and $U_1^{(i)}$ are disposal constants and are not independent as they are linked through the equation of motion and heat conduction. By making use of Eq. (14) stresses and temperature gradient can be expressed as:

$$\bar{\sigma}_{33}=\sum_{l=1}^{4}r_{33(l)}\left(U^{(2l-1)}E_l^+ +U^{(2l)}E_l^-\right) \tag{17}$$
$$\bar{\sigma}_{13}=\sum_{l=1}^{4}r_{13(l)}\left(U^{(2l-1)}E_l^+ +U^{(2l)}E_l^-\right) \tag{18}$$
$$\bar{T}'=\sum_{l=1}^{4}\Omega_l\left(U^{(2l-1)}E_l^+ +U^{(2l)}E_l^-\right) \tag{19}$$

where

$$r_{33(\ell)}=2\mu\alpha_\ell,\ r_{33(2)}=\frac{\mu\left[\alpha_3^2-1\right]}{\alpha_3},$$
$$r_{1(m+1)}=-r_{1(m)},\ m=1,3,5$$
$$r_{13(l)}=\left[\mu\left(\zeta^2/c_2-2\right)\right]$$
$$r_{13(2)}=2\mu,\ r_{3(1)}=r_{3(5)},$$
$$r_{3(m+1)}=r_{3(m)},\ \Omega_\ell=\Theta_\ell^{\,\iota\xi\alpha_\ell}$$
$$\Omega_3=0,\ \Omega_{m+1}=\Omega_m,\ \Omega_3=0,\ \ell=1,3\ \ m=1,3,5 \tag{20}$$

The dispersion relation associated with the plate is now derived from equations by applying traction free boundary conditions:

$$\tau_{13}=\tau_{33}=\frac{\partial T}{\partial x_3}=0$$

at the upper and lower faces $x_3=\pm\frac{d}{2}$ of the plate thus:

$$\sum_{l=1}^{3}r_{33(l)}\left(U^{(2l-1)}e^{i\xi\alpha_l d/2}+U^{(2l)}e^{-i\xi\alpha_l d/2}\right)=0 \tag{21}$$

$$\sum_{l=1}^{3} r_{33(l)}\left(U^{(2l-1)}e^{-i\xi\alpha_l d/2}+U^{(2l)}e^{i\xi\alpha_l d/2}\right)=0 \tag{22}$$

$$\sum_{l=1}^{3} r_{13(l)}\left(U^{(2l-1)}e^{i\xi\alpha_l d/2}-U^{(2l)}e^{-i\xi\alpha_l d/2}\right)=0 \tag{23}$$

$$\sum_{l=1}^{3} r_{13(l)}\left(U^{(2l-1)}e^{-i\xi\alpha_l d/2}-U^{(2l)}e^{i\xi\alpha_l d/2}\right)=0 \tag{24}$$

$$\sum_{l=1}^{3} \Omega_{(l)}\left(U^{(2l-1)}e^{i\xi\alpha_l d/2}-U^{(2l)}e^{-i\xi\alpha_l d/2}\right)=0 \tag{25}$$

$$\sum_{l=1}^{3} \Omega_{(l)}\left(U^{(2l-1)}e^{-i\xi\alpha_l d/2}-U^{(2l)}e^{i\xi\alpha_l d/2}\right)=0 \tag{26}$$

The symmetry of the plate about x_3 =0 allows us to simplify the system of six homogeneous equations in six unknowns into two system of three equations in three unknowns, which then yield the following dispersion relations associated with the plate:

$$\sum_{l=1}^{3} r_{33(l)}\tilde{U}_l^{+}C_l = 0 \tag{27}$$

$$\sum_{l=1}^{3} r_{13(l)}\tilde{U}_l^{+}S_l = 0 \tag{28}$$

$$\sum_{l=1}^{3} \Omega_l\tilde{U}_l^{+}S_l = 0 \tag{29}$$

and

$$\sum_{l=1}^{3} r_{33(l)}\tilde{U}_l^{-}S_l = 0 \tag{30}$$

$$\sum_{l=1}^{3} r_{13(l)}\tilde{U}_l^{-}C_l = 0 \tag{31}$$

$$\sum_{l=1}^{3} \Omega_l\tilde{U}_l^{-}C_l = 0 \tag{32}$$

within which $C_l = cos(\xi\alpha_l\, d/2)$, $S_l = \sin(\xi\alpha_l\, d/2)$ and $\tilde{U}_l^{+} = U^{(2l-1)} + U^{(2l)}$, $\tilde{U}_l^{-} = U^{(2l-1)} - U^{(2l)}$.

The condition that the system of Eqs. (27-29) and (30-32) admit a non-trivial solution give rise to the dispersion relations associated with flexural and extensional waves respectively.

5. AN ASYMPTOTIC ANALYSIS

— Long wavelength

In the numerical section different low wave number limiting behavior was observed for extensional and flexural waves. The limiting behavior of the flexural dispersion relation under a plane strain approximation in which only the fundamental mode retains a finite value as $kd \to 0$ for extensional waves however, both the fundamental mode, the first mode and thermal mode have finite wave speed in this limit, thus giving an additional mode to three long wave limits for the plate in the generalized theory of thermoelasticity.

— Short wavelength

We now seek approximation for the frequencies, as function of wave number, in the long wave high frequency motion regime, that is in the vicinity of the non-zero cutt off frequencies. It is known that for this type of motion: $\tilde{v} \to \infty$ as $kd \to \infty$. Analysis of the relative orders of the coefficients of the secular equation suggests that two roots of the order $O(\tilde{v}^2)$, with the third root of order one. More specifically, approximation for α_1^2, α_2^2 and α_3^2 are taken into account.

6. NUMERICAL DISCUSSIONS AND CONCLUSIONS

In general the waves are dispersive; to discuss the long and short waves, it is required to find numerical solution of the Eqs. (27-29) and (30-32). Computation for the symmetric (flexural) and antisymmetric modes (extensional) have been carried out for an aluminum plate whose physical data are given in Table 1.

Table 1. Physical Properties of an Aluminum Alloy

Young's modulus (E)	72.6 GPa
Poisson ratio (ν)	0.33
Density (ρ)	2800 $kg.m^{-3}$
Specific heat (C_e)	960 $J.kg^{-1}.K^{-1}$
Thermal diffusivity (k_1)	7×10^{-5} $m^2.s^{-1}$
Expansion coefficient (α_t)	2.35×10^{-5} K^{-1}
Initial temperature (T_0)	293 K

The phase and group velocities, ($\tilde{v}$ and $U = \tilde{v} + k\dfrac{d\tilde{v}}{dk}$, respectively) dispersion curves, are plotted as a function of the wave-number assuming the thickness d of the plate is fixed.

These curves have been calculated from expression based on the dispersion relation in Eqs. (27-29) and (30-32), which are decoupled characteristic equations corresponding to symmetric and anti-symmetric modes of vibrations in LS theory of generalized thermoelasticity. The additional new mode to those already observed in purely elastic materials is the quasi-thermal T-mode. Dispersion curves for symmetric and antisymmetric modes in LS theory of generalized thermoelasticity are shown in Figs. 1-4, the various modes get merged and then approach each other as wave number increases, where the phase and group velocities tend towards the Rayleigh surface wave speed.

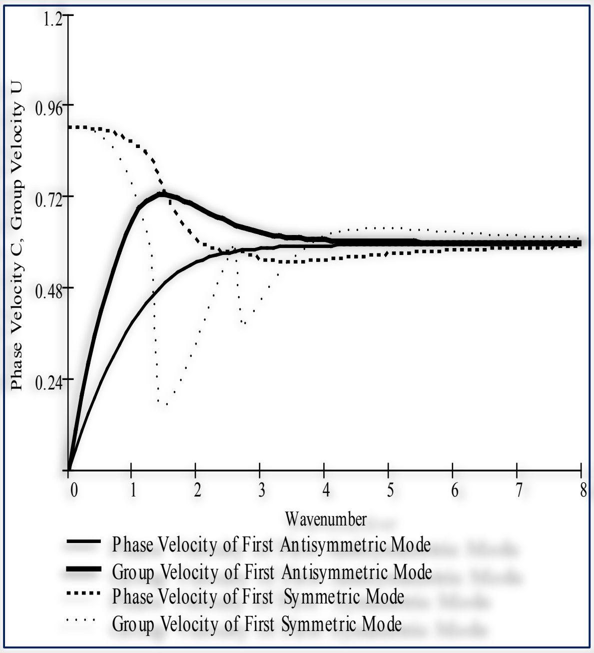

Figure 1. Dispersion curves of first symmetric and antisymmetric modes in the LS Theory

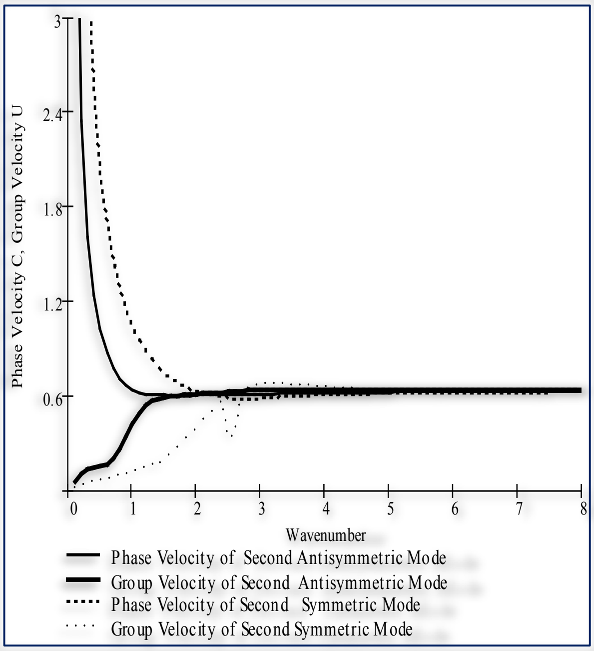

Figure 2. Dispersion curves of second symmetric and antisymmetric modes in the LS Theory

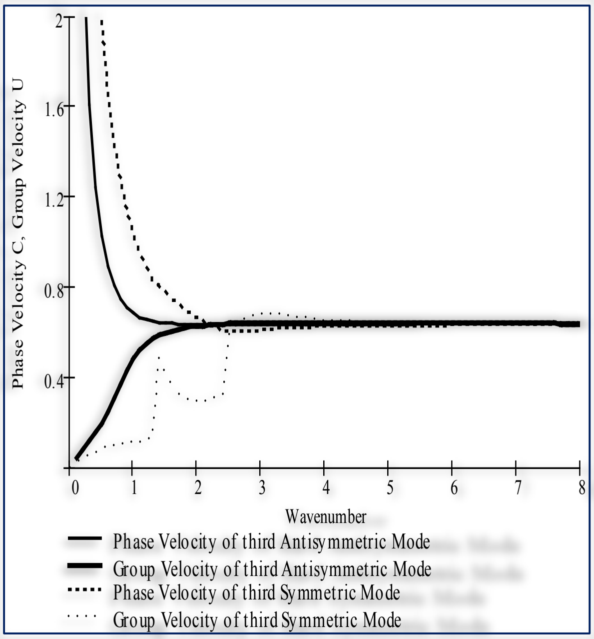

Figure 3. Dispersion curves of third symmetric and antisymmetric modes in the LS Theory

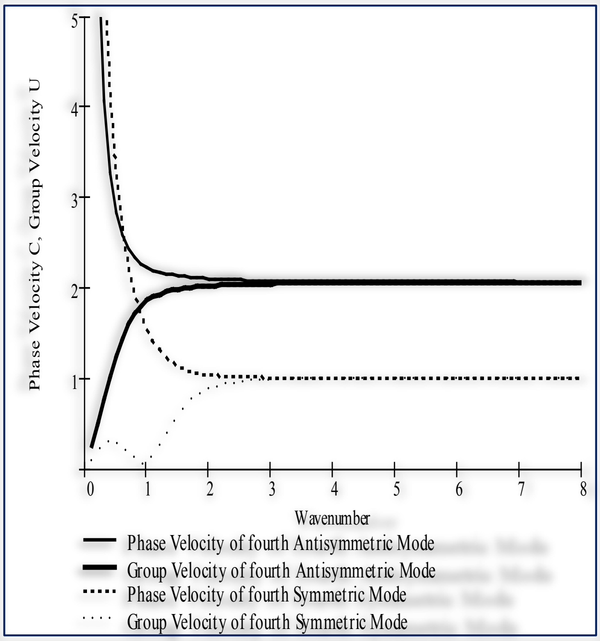

Figure 4. Dispersion curves of fourth symmetric and antisymmetric modes in the LS Theory

The wave modes are observed to be more affected at the zero wave number limits, due to the thermomechanical effects. When the thermal relaxation time is zero, then the results obtained in the analysis reduce to the conventional coupled theory of thermo-elasticity. When the coupling constant is identically equal to zero, the strain and thermal fields are uncoupled from each other. In this case the results can be obtained from the uncoupled theory of thermoelasticity.

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