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THERMOELASTIC ANALYSIS OF RADIALLY GRADED SPHERICAL PRESSURE VESSELS

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Abstract: This paper deals with the thermoelastic analysis of functionally graded hollow spherical bodies subjected to constant mechanical and thermal loading. The temperature field is arbitrary function of the radial coordinate, the material properties vary according to power law functions along the radius of the sphere. An analytical method is presented to determine the displacements and stresses within the spherical body. The method is expanded to tackle the problem of spherical bodies made from temperature dependent radially graded materials. The results are compared to results coming from finite element simulations.
Keywords: FGM, spherical body, thermal stresses, FEM

1. INTRODUCTION

In recent years functionally graded materials (FGMs) have been widely used in numerous engineering applications due to their excellent mechanical properties. The smooth transition between the components decreases the chance of debonding and cracking under mechanical and thermal loads. Lots of books and papers deal with the mechanics of FGMs from different aspects. Books by Boley and Weiner [1], Nowinski [2], Lekhnitskii [3] and Shen [4] give solutions to many linearly elastic problems for non-homogeneous components. There are a lot of papers such as [5-12] that investigates different thermoelastic problems of heterogeneous simple structural components (disks, beams, cylinders or plates).

Zimmerman and Lutz ([13], [14]) gave analytical solutions for the stresses and displacements in radially graded spherical bodies and circular cylinders. Paper [15] presented solutions for thick-walled radially graded cylinders and spheres where the material properties was described by exponential functions. Kar and Kanoria [16] dealt with the determination of thermo-elastic interaction due to a step thermal loading on the boundary surfaces of a radially graded orthotropic sphere in the context of linear theories of generalized thermoelasticity. Bich and Tung [17] presented an analytical approach to investigate the non-linear axisymmetric response of radially graded shallow spherical shells subjected to constant external pressure and temperature field incorporating the effects of imperfections. Paper [18] studied the elastic and perfectly plastic radially graded spheres where the material properties were power functions of the radial coordinate. Nayak et. al and Bayat et. al in [19] and [20] presented analytical solutions to obtain the thermal stresses within thick spherical pressure vessels made of FGMs subjected to axisymmetric thermomechanical loads. The material properties are assumed to be graded in the radial direction based on the power-law function of the radial coordinate but the Poisson ratio has constant value and the temperature fields had a specific forms. Gönczi in [21] and [22] derived analytical methods to calculate the thermal stresses in multilayered spheres and disks, then used these methods to deal with the general problems of radially graded spherical pressure vessels and disk with arbitrary temperature dependent material properties. Ye et. al [23] studied the vibration of laminated functionally graded spherical shells with general boundary conditions assuming power law distribution based on three-dimension linearized shell theory and Rayleigh-Ritz method.

Nematolli et. al. [24] gave an analytical solution for the displacement and stress fields in thick-walled rotating spherical pressure vessel made of functionally graded materials in a uniform magnetic field assuming power law distribution along the thickness.

In paper [25] a numerical method is presented to determine the thermoelastic wave in multilayered spherical shells with functionally graded layers under thermal loading based on Lord-Shulman generalized coupled thermoelasticity theory.

Viola et. al. [26] developed a numerical method for the static behavior of functionally graded spherical shells and panels subjected to uniform loading, the material properties were power law functions of the thickness coordinate. Qoliha and Fadaee [27] dealt with the axisymmetric bending analysis of a hybrid piezoelectric layered functionally graded spherical cap. The analytical solution was given based on thin shell theory and coupled electro-mechanical equations. Arefi and Zenkour [28] studied the problems of functionally graded spherical pressure vessels using non-linear shell theory and Adomians decomposition method. In [29] a closed form analytical solution is presented for special thermoelastoplastic problems of thick-walled spheres. Akinlabi et. al. [30] developed a thermoelastoplastic method to calculate the stresses and displacements in functionally graded spheres after thermal treatment. Paper [31] used perturbation technique to solves incompressible spherical shell problems with temperature dependent material properties.

Our aim is to present an analytical solution for radially graded spheres subjected to combined mechanical and thermal loads. As we can see in the literature, the temperature fields have specific forms and the temperature dependency is often neglected. In this paper the temperature field T is arbitrary function of the radial coordinate, the Poisson's ratio is constant, while the material properties are power functions of the radial coordinate:

$$E(r) = E_0\left(\frac{r}{R_1}\right)^m, \ \alpha(r) = \alpha_0\left(\frac{r}{R_1}\right)^m, \ T = T(r). \tag{1}$$

At first a radially graded hollow sphere is considered, the sketch of the problem can be seen in Figure 1 where a spherical coordinate system is used. Our aim is to determine the analytical solution for the stress and displacement fields within the components, then develop a multilayered approach (with n sublayers) to tackle the temperature dependency of the material properties, where the internal radius is $R_1=a$, the outer radius is denoted by $b(=R_{n+1})$.

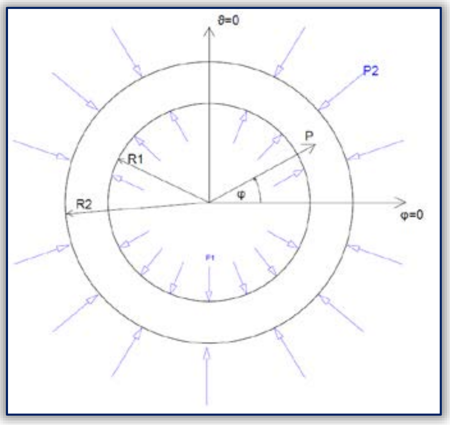

Figure 1. The sketch of the problem

2. GOVERNING EQUATIONS

In our case displacement field has one non-zero coordinate u_r. The kinematic equations can be expressed as

$$\varepsilon_r = \frac{\mathrm{d}u_r}{\mathrm{d}r}, \tag{2}$$

$$\varepsilon_\varphi = \varepsilon_\vartheta = \frac{u_r}{r}, \quad a \le r \le b, \tag{3}$$

where ε_r, ε_φ, ε_ϑ denote the normal strain coordinates of the strain tensor. The stress-strain relations lead to

$$\sigma_r = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_r + 2\nu\varepsilon_\varphi - (1+\nu)\alpha T\right],$$
$$\sigma_r = E_\nu\left[(1-\nu)\frac{\mathrm{d}u_r}{\mathrm{d}r} + 2\nu\frac{u_r}{r} - (1+\nu)\alpha_0\left(\frac{r}{a}\right)^m T\right], \tag{4}$$

$$E_\nu = \frac{E_0\left(\frac{r}{a}\right)^m}{(1+\nu)(1-2\nu)}, \tag{5}$$

$$\sigma_\varphi = \sigma_\theta = \frac{E}{(1+\nu)(1-2\nu)}\left[\nu\varepsilon_r + \varepsilon_\varphi - (1+\nu)\alpha T\right],$$
$$\sigma_\varphi = \sigma_\theta = E_\nu\left[\nu\frac{\mathrm{d}u_r}{\mathrm{d}r} + \frac{u_r}{r} - (1+\nu)\alpha_0\left(\frac{r}{a}\right)^m T\right], \tag{6}$$

where ν denotes Poisson's ratio, σ_r, σ_φ and σ_ϑ are the radial and hoop (or tangential normal) stresses. The equilibrium equation in this case can be given as

$$\frac{\mathrm{d}\sigma_r}{\mathrm{d}r} + \frac{2}{r}\left(\sigma_r - \sigma_\varphi\right) + b_r = 0, \tag{7}$$

where b_r is the radial body force.
The combination of Eqs. (7), (4) and (6) yields to a differential equation for the radial displacement field. When the body force is neglected, the solution is

$$
\begin{aligned}
u(r) = {} & C_2 r^{A_1} + C_1 r^{A_2} - \frac{2\alpha_0(1+v)}{G_1}\Bigg\{\Bigg[-\frac{1}{2} r^{\frac{-G_1+(m+1)(1-v)}{2(v-1)}}\left(\int_a^r r^{\frac{+G_1+(m+3)(v-1)}{2(v-1)}}\left(\frac{r}{a}\right)^m \frac{\mathrm{d}}{\mathrm{d}r}T(r)\mathrm{d}r\right) \\
& + \frac{1}{2} r^{\frac{G_1+(m+1)(1-v)}{2(v-1)}}\left(\int_a^r r^{\frac{-G_1+(m+3)(v-1)}{2(v-1)}}\left(\frac{r}{a}\right)^m \frac{\mathrm{d}}{\mathrm{d}r}T(r)\mathrm{d}r\right) + \\
& + m\left[r^{\frac{G_1+(m+1)(1-v)}{2(v-1)}}\left(\int_a^r r^{\frac{-G_1+(m+1)(v-1)}{2(v-1)}}\left(\frac{r}{a}\right)^m T(r)\mathrm{d}r\right) - r^{\frac{-G_1+(m+1)(1-v)}{2(v-1)}}\left(r^{\frac{G_1+(m+1)(v-1)}{2(v-1)}}\left(\frac{r}{a}\right)^m T(r)\mathrm{d}r\right)\right]\Bigg]\Bigg\},
\end{aligned}
\tag{8}
$$

$$
A_1 = \frac{1}{2}\frac{(m+1)(1-v)+G_2}{v-1}, \quad A_2 = -\frac{1}{2}\frac{(m+1)(1-v)+G_2}{v-1}. \tag{9}
$$

With a constant radial body force, the general solution for the displacement field can be expressed as

$$
\begin{aligned}
u(r) = {} & C_2 r^{\frac{1}{2}\frac{(m+1)(1-\nu)+G_2}{\nu-1}} + C_1 r^{-\frac{1}{2}\frac{(m+1)(1-\nu)+G_2}{\nu-1}} - \\
& -\frac{1}{G_1}\Bigg\{2\alpha_0(1+\nu)\Bigg[-\frac{1}{2} r^{\frac{-G_1+(m+1)(1-\nu)}{2(\nu-1)}}\left(\int_a^r r^{\frac{+G_1+(m+3)(\nu-1)}{2(\nu-1)}}\left(\frac{r}{a}\right)^m \frac{\mathrm{d}}{\mathrm{d}r}T(r) + b_r\left(v-\frac{1}{2}\right)\left(\frac{r}{a}\right)^{-m}\mathrm{d}r\right) + \\
& + \frac{1}{2} r^{\frac{G_1+(m+1)(1-\nu)}{2(\nu-1)}}\left(\int_a^r r^{\frac{-G_1+(m+3)(\nu-1)}{2(\nu-1)}}\left(\frac{r}{a}\right)^m \frac{\mathrm{d}}{\mathrm{d}r}T(r)\mathrm{d}r\right) + \\
& + m\left[\begin{aligned} & r^{\frac{G_1+(m+1)(1-\nu)}{2(\nu-1)}}\left(\int_a^r r^{\frac{-G_1+(m+1)(\nu-1)}{2(\nu-1)}}\left(\frac{r}{a}\right)^m T(r)\mathrm{d}r\right) - \\ & -r^{\frac{-G_1+(m+1)(1-\nu)}{2(\nu-1)}}\left(b_r(-1+2v)\left(\frac{r}{a}\right)^{-m} + r^{\frac{G_1+(m+1)(\nu-1)}{2(\nu-1)}}\left(\frac{r}{a}\right)^m T(r)\mathrm{d}r\right)\end{aligned}\right].
\end{aligned}
\tag{10}
$$

With the displacement field, the normal stresses can be calculated using Eqs. (4-6). The next step is the determination of the unknown constants of integration C_1 and C_2. When the traction boundary conditions are given, we have:

$$
\sigma_r(r=a) = -p_1, \quad \sigma_r(r=b) = -p_2. \tag{11}
$$

3. TEMPERATURE DEPENDENT MATERIAL

Next we consider a temperature dependent radially graded material. Only the Young modulus, Poisson's ratio and the coefficient of linear thermal expansion are arbitrary functions of the temperature field and are power functions of the radial coordinate r (except the Poisson's ratio), e.g.:

$$
\begin{aligned}
& E(T,r) = f(T)E_0\left(\frac{r}{a}\right)^m, \\
& \alpha(T,r) = g(T)\alpha_0\left(\frac{r}{a}\right)^m, \\
& \nu(T) = h(T).
\end{aligned}
\tag{12}
$$

Here we will use multilayered approach. The body is divided into sublayers, where in the i-th layer we have:

$$
u_{r,i}(r),\ \sigma_{r,i}(r),\ \sigma_{\varphi,i}(r), \qquad R_i \le r \le R_{i+1}. \tag{13}
$$

The temperature field is given, the sphere is divided into n sublayers, where we use discrete values of the temperature field for every layer. For example, we can use temperature value at the middle of each layer $T_{a,i} = T(r = (R_i + R_{i+1})/2)$, which means, that

$$
\begin{aligned}
& E(T = T_{a,i}, r) = f(T_{a,i})E_0\left(\frac{r}{a}\right)^m = E_{0,i}\left(\frac{r}{a}\right)^m, \\
& \alpha(T = T_{a,i}, r) = g(T_{a,i})\alpha_0\left(\frac{r}{a}\right)^m = \alpha_{0,i}\left(\frac{r}{a}\right)^m, \\
& \nu(T = T_{a,i}) = h(T_{a,i})\nu = \nu_i.
\end{aligned}
\tag{14}
$$

With this formulation we have the same solutions as in Eqs. (4-9). For every layer, we have 2 unknown constants, which means that we need $2n$ equations to calculate them. There are two fitting conditions between the adjacent layers:

$$\begin{aligned}&\sigma_{r,i}(r=R_{i+1})=\sigma_{r,i+1}(r=R_{i+1}),\\&u_{r,i}(r=R_{i+1})=u_{r,i+1}(r=R_{i+1}),\ i=1,..n-1.\end{aligned}\qquad(15)$$

which ensures the continuity of the radial stress and displacement fields. Furthermore, we have two additional boundary conditions at the boundary surfaces of the whole spherical body - Eq. (11). This way the more layers we have the more accurate the solution will be.

4. NUMERICAL EXAMPLES

For the first numerical example the following numerical data will be used:

$$R_1=a=1\text{m},\ b=1.1\text{m},\ m=2.3,\ \nu=0.3,\ E_0=2\cdot10^{11}\frac{\text{N}}{\text{m}^2},\ \alpha_0=1.2\cdot10^{-6}\frac{1}{\text{K}},$$

$$T(r)=-980+1000r^3\left[{}^\circ C\right],\ \ p_1=80\,\text{MPa},\ p_2=0\text{MPa}.$$

The previously developed method will be implemented in Maple, the finite element simulations will be carried out by Abaqus. In the finite element model axisymmetric formulation is used, with quadratic coupled temperature-displacement elements. Only a quarter of the geometry is modeled and symmetry boundary conditions are used. The radially graded body is modelled as multilayered sphere with 20 homogeneous, perfectly bonded layers. The finite element model, the mesh and the hoop stress distribution can be seen in Figure 2.

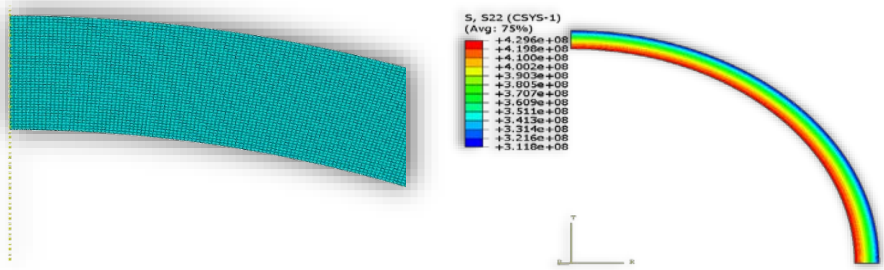

Figure 2. The finite element model of the sphere with the hoop stress

The displacement fields coming from the analytical and numerical methods are plotted in Figure 3. The stress distributions can be seen in Figures 4 and 5.

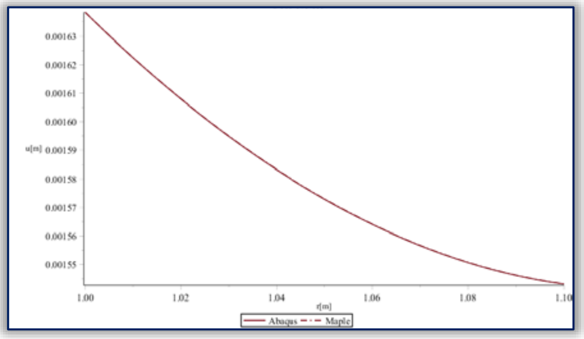

Figure 3. The displacement fields of the analytical and numerical methods

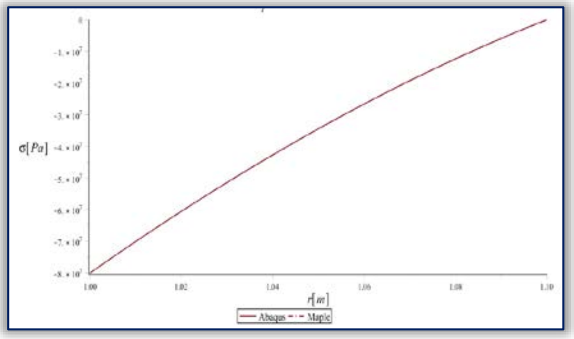

Figure 4. The radial stress distributions of the analytical and numerical methods

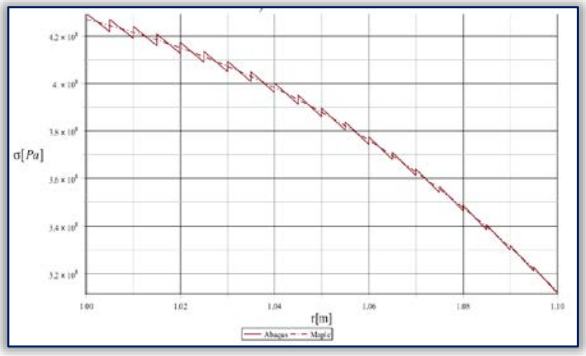

Figure 5. The hoop stresses coming from the finite element simulations and the analytical solution

In our second numerical example the temperature dependency of the material is considered. The following data will be used:

$$R_1 = a = 0.04,\ b = 0.06,\ p_1 = 80\,\mathrm{MPa},\ p_2 = 0\mathrm{MPa}, E_0 = 2\cdot 10^{11},\ \alpha_0 = 1.2\cdot 10^{-6},$$

$$E(r,T) = E_0(1.2 + 2\cdot 10^{-5}T^2)\left(\frac{r}{R_1}\right)^m,\ \ \alpha(r,T) = \alpha_0(1.2 + 2\cdot 10^{-5}T^2)\left(\frac{r}{R_1}\right)^m,$$

$$\nu(T) = 0.3(1.2 + 2\cdot 10^{-5}T^2),\ T(r) = 40 + 500000r^3\left[{}^\circ C\right],\ m = 2.3.$$

Let the number of layers be n=50, the stress fields are plotted in Figure 6. To eliminate the oscillation of the tangential normal (or hoop) stress coming from the sublayer technique, we used the curve fitting method presented in [21]. The results were in good agreement with the finite element simulations.

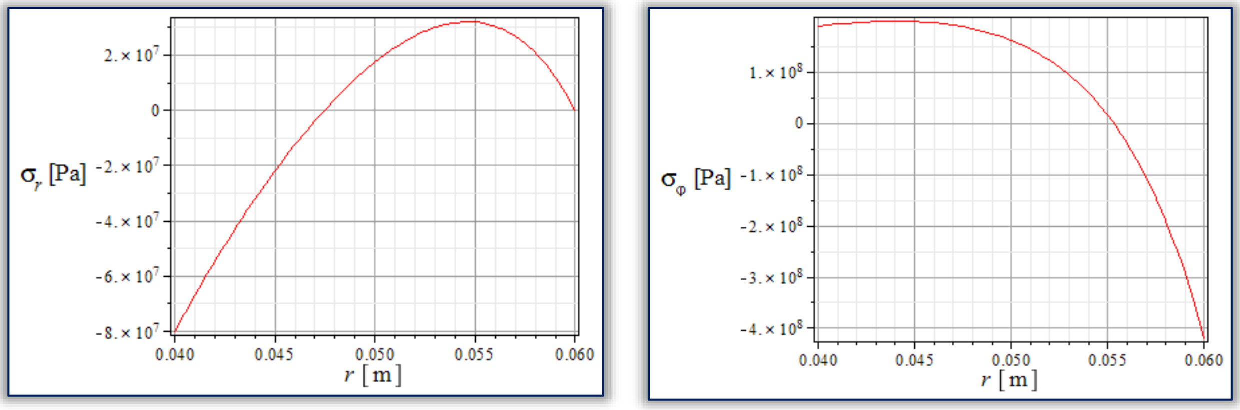

Figure 6. The radial and tangential normal stresses of the second numerical example

5. CONCLUSIONS

An analytical method was presented to determine the stresses and displacements in radially graded spherical pressure vessels. The material properties were assumed to be power law functions of the radial coordinate. The temperature field was arbitrary function of the radial coordinate. We used these equations to tackle the temperature dependency of the material using multilayered approach. The results of the developed methods were verified by finite element simulations.

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