[1.] Jyoti Chetan VANIKAR, [2.] Rajeev Kumar BANSAL, [3.] Vineeta BASOTIA

APPROXIMATION OF SUBSURFACE SEEPAGE FLOW OVER SLOPING TERRAIN IN A DITCH DRAIN SYSTEM UNDER CONSTANT RECHARGE

[1, 3.] JJT University, Jhunjhunu, Rajasthan, INDIA
[2.] National Defence Academy, Khadakwasla, Pune, Maharashtra, INDIA

Abstract: New analytical solutions are developed and illustrated to estimate water table variations in combined response to subsurface seepage and downward percolation in a sloping unconfined porous medium. A nonlinear advection-diffusion equation characterizing groundwater flow in a ditch-drain aquifer system is subjected to time dependent boundary conditions is considered. Approximate analytical solutions presented in the study are based on linearization of the Boussinesq equation and are capable of predicting spatio-temporal distribution of the water head and discharge rate at the stream-aquifer interfaces. Validation of linearization is assessed by solving the nonlinear advection-diffusion equation using Du Fort and Frankel scheme. A simple and efficient iterative scheme is also developed for computation the groundwater mound height and its spatial location. Sensitivity of various aquifer parameters is analyzed.
Keywords: Ditch-drain; Sloping aquifer; Water table; Boussinesq equation

1. INTRODUCTION

Unconfined aquifer acts as an interesting object in ground water hydrology because of a free surface, often referred to as water table. Numerous mathematical models have been developed and analyzed by researchers in past two decades for prediction of water table under dynamic boundary conditions. The unconfined flow of ground water is approximated by a nonlinear partial differential equation, known as the Boussinesq equation. Its analytical solution is useful for validation of numerical models and serves as guidelines for experimental works. Boussinesq was the first one to consider one dimensional flow and obtain an exact solution by using the method of separation of variables. By adopting Power series method Polubarinova-Kochina (1962) obtain a solution and represented it in graphical form. Some more remarkable works in this direction includes Baumann (1952), Dagan (1967), Hantush (1967) and Rao and Sharma (1981). An analytical solution of linearized version of Boussinesq equation was derived by Cooper and Rorabaugh (1963), Dicker and Sevian (1965), Rai and singh (1995). An approximate polynomial solution was presented by Tolikas (1984). Besides analytical solution, numerous numerical solutions have been presented by researchers such as Yeh (1970), Hornberger et al (1970), Lin (1972), Teloglou and Bansal (2012).

Many hydrologists developed approximations of time varying property of recharge rate using field data (Zomordi, 1991; Bansal and Teloglou, 2013). A new scheme using a sequence of line segments of various length and slope for the simulation entire recharge was proposed by Manglik et al (1997) for single as well as multiple recharge basins. Approximations of recharge rate by considering an exponential decaying function of time was used by Rai and Singh (1996), Chang and Yeh (2007), Bansal (2012). Teloglou et al (2008) approximated the time varying recharge by using polynomial function. The transient behavior with or without vertical recharge of stream aquifer seepage models have been presented by Bansal and Das (2009, 2010, 2011). Analytical solution of three dimensional saturated-unsaturated flow influenced by localized recharge in unconfined aquifers was depicted by Chang and Yeh (2017).

Water table fluctuations induced due to transient recharge from circular basin was studied by Teloglou et al (2008) by considering aquifers overlying by semi-impervious layers. Ireson and Butler (2013) critically assessed simple recharge models. van der Spek et al (2013) provided a thorough picture of characterization of groundwater dynamics in landslides in varved clays, Yeh and Chang (2013) focused on future requirement of research through their work. Analytical solution of three dimensional saturated unsaturated flow influenced by localized recharge in unconfined aquifers was depicted by Chang and Yeh (2017). Boyraz and Kazezyilmaz-Alhan (2018) presented analytical and experimental model for the solution of a two dimensional groundwater flow in stream aquifer interaction with sloping stream boundary. To simulate this aquifer-stream system experiments were conducted. Verification of analytical and experimental results was analyzed by using Visual MODFLOW.

The major objective of the present work is to develop new analytical expressions for water head and flow rate in a sloping ditch-drain aquifer system. The mathematical model consists of an unconfined sloping aquifer of finite length, which is in contact with surface water of in two adjacent ditches. The water level in the ditches varies from a known initial value to a final value by an exponentially decaying function of time. Besides seepage from the ditches, the aquifer also receives a uniform recharge throughout its domain. The subsurface seepage model is simulated using a nonlinear Boussinesq equation based on extended Dupuit-Forchheimer assumptions. Linearized version of Boussinesq equation is then solved by employing Laplace transform method. Efficiency of linearization method is assessed by solving nonlinear Boussinesq equation using Du Fort Frankel scheme. Some special cases such as upward slope, zero slope and abrupt rise in the water level are deduced directly from the main result. Analytical results are illustrated with the help of a numerical example. Impacts of various hydrological parameters such as bed slope of the aquifer, recharge rate of transient water table are illustrated.

2. DEVELOPMENT OF MODEL AND ANALYTICAL SOLUTION

Schematic diagram of the model domain is given in Figure 1. An unconfined aquifer with downward slope tan β is hydrologically contacted with two ditches or streams with initial water level as h_0. Water level in the left stream rises from its initial level h_0 to a known value h_L by an exponentially decaying function of time t. Similarly, water level in the right end stream rises from h_0 to h_R at different rate. Furthermore, the aquifer is replenished vertically with constant downward recharge. The analytical model is based on the following assumptions:

(a) aquifer is underlain by a completely impervious sloping bed,
(b) variations in the water table are small compared to its initial value,
(c) variations in the hydraulic conductivity and specific yield of the aquifer with spatial coordinate are negligible,
(d) the stream bank is nearly vertical and penetrates the full depth of the aquifer, and
(e) streamlines are nearly parallel to the sloping base (extended Dupuit-Forchheimer assumption).

The groundwater flow in sloping unconfined aquifer is approximated by a nonlinear Boussinesq equation (Childs, 1971; Chapman, 1980; Bansal, 2013)

$$K\cos^2\beta\frac{\partial}{\partial x}\left(h\frac{\partial h}{\partial x}\right) - K\sin\beta\cos\beta\frac{\partial h}{\partial x} + N = S\frac{\partial h}{\partial t} \tag{1}$$

where K is the hydraulic conductivity, S is specific yield of the aquifer and h(x, t) is the height of water head measured along vertical direction from the sloping bed. N is constant recharge rate. Since the streamlines are considered to be parallel to the sloping bed, the initial water table in the aquifer can be assumed to be constant and same as that of the initial height of water level in the streams, i.e.

$$h(x, t = 0) = h_0 \tag{2a}$$

Water in the left stream rises from level h_0 to a final level h_L by an exponentially decaying function of time, given by

$$h(x = 0, t > 0) = h_L - (h_L - h_0)e^{-\lambda_1 t} \tag{2b}$$

where λ_1 is the parameter controlling the rise rate in the left stream. In a similar manner, the boundary condition at the right end of the aquifer is given by

$$h(x = L, t > 0) = h_R - (h_R - h_0)e^{-\lambda_2 t} \tag{2c}$$

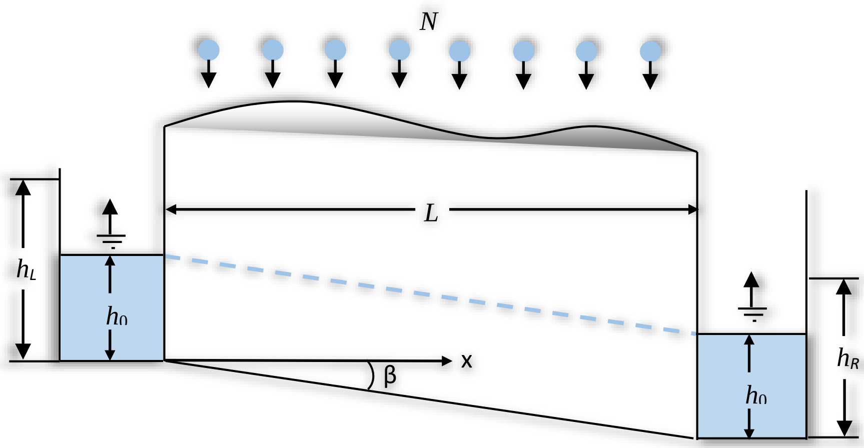

Figure 1: Definition sketch of the stream-aquifer model

Equation (1) is a nonlinear parabolic partial differential equation which does not admit general solution. Approximate analytical solution of equation (1) can be derived by linearizing it using technique of Marino (1973) and solving it by standard technique. Rewrite equation (1) as

$$KD\cos^2\beta\frac{\partial^2 h}{\partial x^2} - K\sin\beta\cos\beta\frac{\partial h}{\partial x} + N = S\frac{\partial h}{\partial t} \tag{3}$$

where D is the average saturated thickness of the aquifer, and is given by $D = (h_0 + h_t)/2$ where h_0 is the initial water level in the aquifer and h_t is the height at the current time t (Marino, 1973). Now, define following dimensionless variables

$$H = \frac{h - h_0}{L};\ X = \frac{x}{L};\ \tau = \frac{K\,D\,\cos^2\beta}{SL^2}t \tag{4}$$

where L is the length of the domain. Equation (1) now translates to

$$\frac{\partial^2 H}{\partial X^2} - 2\alpha\frac{\partial H}{\partial X} + N' = \frac{\partial H}{\partial \tau} \tag{5}$$

where $\alpha = (L\tan\beta)/(2D)$ and $N' = (N\,L)/(K\,D\cos^2\beta)$. The initial and boundary conditions become

$$H(X, \tau = 0) = 0 \tag{6a}$$

$$H(X = 0, \tau) = h'_L\left(1 - e^{-\lambda'_1\tau}\right) \tag{6b}$$

$$H(X = 1, \tau) = h'_R\left(1 - e^{-\lambda'_2\tau}\right) \tag{6c}$$

where $h_L' = (h_L - h_0)/L$ and $h_R' = (h_R - h_0)/L$. Dimensionless forms of the parameters controlling stream rise rate are

$$\lambda'_1 = \frac{SL^2}{K\,D\,\cos^2\beta}\lambda_1;\ \lambda'_2 = \frac{SL^2}{K\,D\,\cos^2\beta}\lambda_2 \tag{7}$$

Equation (5) subject to conditions (6a) – (6c) can be solved using Laplace transform. Define

$$L\{H(X,\tau)\} = \varphi(X,s) = \int_0^\infty H(X,\tau)\,e^{-s\tau}d\tau \tag{8}$$

Taking Laplace transform of equation (5), one gets

$$\frac{d^2\varphi}{dX^2} - 2\alpha\frac{d\varphi}{dX} - s\varphi = -N' \tag{9}$$

Equation (9) can be solved by any standard method. Its general solution is

$$\varphi(X,s) = e^{\alpha X}\left\{A\cosh\left(X\sqrt{\alpha^2+s}\right) + B\sinh\left(X\sqrt{\alpha^2+s}\right)\right\} + \frac{N'}{s^2} \tag{10}$$

where A and B are arbitrary constants. In order to determine A and B, we apply Laplace transform to the boundary conditions (6b) and (6c) and use them in equation (8). The resulting expressions are

$$A = \frac{h'_L\lambda'_1}{s(s+\lambda'_1)} - \frac{N'}{s^2} \tag{11}$$

$$B = \frac{\left\{\frac{h'_R\lambda'_2}{s(s+\lambda'_2)} - \frac{N'}{s^2}\right\}e^{-\alpha} - \frac{h'_L\lambda'_1\cosh\sqrt{\alpha^2+s}}{s(s+\lambda'_1)} + \frac{N'\cosh\sqrt{\alpha^2+s}}{s^2}}{\sinh\sqrt{\alpha^2+s}} \tag{12}$$

Substituting values of A and B in equation (10), one obtains

$$\varphi(X,s) = \frac{e^{\alpha X}\left\{\frac{h'_R\lambda'_2}{s(s+\lambda'_2)} - \frac{N'}{s^2}\right\}\sinh\{(1-x)\sqrt{\alpha^2+s}\}}{\sinh\sqrt{\alpha^2+s}} + \frac{e^{-\alpha(1-X)}\left\{\frac{h'_R\lambda'_2}{s(s+\lambda'_2)} - \frac{N'}{s^2}\right\}\sinh\{x\sqrt{\alpha^2+s}\}}{\sinh\sqrt{\alpha^2+s}} + \frac{N'}{s^2} \tag{13}$$

Inverse Laplace transform of equation (13) can be performed using calculus of residue. After some simplification, the solution becomes

$$\begin{aligned}
H(X,\tau) = {} & e^{\alpha X}h'_L\left\{\frac{\sinh\{(1-X)\sqrt{\alpha^2-\lambda'_1}\}}{\sinh\sqrt{\alpha^2-\lambda'_1}}\left(1-e^{-\lambda'_1\tau}\right)\right.\\
& \left. - \sum_{n=1}^{\infty}\frac{2n\pi\sin(n\pi X)}{(\alpha^2+n^2\pi^2)\left(\frac{\alpha^2+n^2\pi^2}{\lambda'_1}-1\right)}\left(1-e^{-(\alpha^2+n^2\pi^2)\tau}\right)\right\}\\
& + e^{-\alpha(1-X)}h'_R\left\{\frac{\sinh\{X\sqrt{\alpha^2-\lambda'_2}\}}{\sinh\sqrt{\alpha^2-\lambda'_2}}\left(1-e^{-\lambda'_2\tau}\right)\right.\\
& \left. + \sum_{n=1}^{\infty}\frac{2n\pi(-1)^n\sin n\pi X}{(\alpha^2+n^2\pi^2)\left(\frac{\alpha^2+n^2\pi^2}{\lambda'_2}-1\right)}\left(1-e^{-(\alpha^2+n^2\pi^2)\tau}\right)\right\}\\
& + e^{\alpha X}N'\sum_{n=1}^{\infty}\frac{2n\pi\{1-(-1)^n e^{-\alpha}\}\sin(n\pi X)}{(\alpha^2+n^2\pi^2)^2}\left(1-e^{-(\alpha^2+n^2\pi^2)\tau}\right)
\end{aligned} \tag{14}$$

Equation (14) provides closed form expressions for the water table height in the unconfined sloping aquifer due to seepage from adjacent ditches/streams and uniform downward percolations. As far as convergence of various infinite series in the right hand side is concerned, it can be easily derived that most of these terms are Fourier sine series expansion of known functions. Few important results in this connection are listed below:

$$\frac{\sinh\{(1-X)\sqrt{\alpha^2-\lambda_1'}\}}{\sinh\sqrt{\alpha^2-\lambda_1'}}-\frac{\sinh\{\alpha(1-X)\}}{\sinh\alpha}=\sum_{n=1}^{\infty}\frac{2n\pi\sin(n\pi X)}{(\alpha^2+n^2\pi^2)\left(\frac{\alpha^2+n^2\pi^2}{\lambda_1'}-1\right)} \tag{15a}$$

$$-\frac{\sinh\{X\sqrt{\alpha^2-\lambda_2'}\}}{\sinh\sqrt{\alpha^2-\lambda_2'}}+\frac{\sinh(\alpha X)}{\sinh\alpha}=\sum_{n=1}^{\infty}\frac{2n\pi(-1)^n\sin(n\pi X)}{(\alpha^2+n^2\pi^2)\left(\frac{\alpha^2+n^2\pi^2}{\lambda_2'}-1\right)} \tag{15b}$$

$$\frac{X\sinh\alpha\cosh\alpha X-\sinh\alpha X\cosh\alpha}{2\alpha(\sinh\alpha)^2}=\sum_{n=1}^{\infty}\frac{2n\pi(-1)^n\sin(n\pi X)}{(\alpha^2+n^2\pi^2)^2} \tag{15c}$$

$$\frac{X\sinh\alpha\cosh\{\alpha(1-X)\}-\sinh\alpha X}{2\alpha(\sinh\alpha)^2}=\sum_{n=1}^{\infty}\frac{2n\pi\sin(n\pi X)}{(\alpha^2+n^2\pi^2)^2} \tag{15d}$$

Numerical experiments indicate that the infinite series in the right-hand side are fast converging and first 50 terms provide fair approximation of the whole series.

3. STEADY STATE PROFILES OF WATER HEAD

As a matter of verification of correctness of analytical solution, it is important to list out the behaviour of water table for large value of time, i.e. steady-state solution. Setting $t \to \infty$ in equation (14) and simplify it to get

$$H(X)=\frac{(h_L' e^{\alpha}-h_R' e^{-\alpha})+e^{-\alpha(1-2X)}(h_R'-h_L')}{2\sinh\alpha}+N' e^{\alpha X}\sum_{n=1}^{\infty}\frac{2n\pi\{1-(-1)^n e^{-\alpha}\}\sin n\pi X}{(\alpha^2+n^2\pi^2)^2} \tag{16}$$

Equation (16) implies that the steady-state value is dependent on bed slope as well as downward recharge rate. The stream-stage parameters λ_1 and λ_2 don't play any deterministic role in it. If the aquifer bed is perfectly horizontal, the steady-state profile of water head becomes

$$H(X)=h_R'-X(h_R'-h_L')+\frac{1}{2}N'X(1-X) \tag{17}$$

which indicate that the free surface attains a parabolic (opening downward) shape. In the above derivation, Fourier sine expansion of the term $X(1-X)$ has been used. It can be easily verified that

$$\frac{X(1-X)}{2}=\sum_{n=1}^{\infty}\frac{2\sin n\pi X}{n^3\pi^3}-\sum_{n=1}^{\infty}\frac{2(-1)^n\sin n\pi X}{n^3\pi^3} \tag{18}$$

The ordinate of its vertex characterizes the height at which the groundwater mound eventually settles. It can be shown that the peak of mound occurs at $X_a = \{1/2 - (h_R'-h_L')/N'\}$. If this height is denoted by H_a, then

$$H_a=h_R'+\frac{N'}{8}\left\{1-\frac{2(h_R'-h_L')}{N'}\right\}^2 \tag{19}$$

In case of zero recharge, the final profiles of water head can be obtained by setting $N' = 0$ in equation (17), yielding

$$H(X)=h_R'-X(h_R'-h_L') \tag{20}$$

which is nothing but a straight line in dimensionless form joining points $(0, h_L)$ and (L, h_R). A special case of equation (19) is for $h_L = h_R$ in which the mound height reduces to

$$H_a=h_R'+\frac{N'}{8} \tag{21}$$

4. DETERMINATION OF DISCHARGE RATE AT THE INTERFACES

Discharge rate per unit area of the aquifer is defined as follows (Chapman, 1980):

$$q(x,t)=-Kh\cos^2\beta\left(\frac{\partial h}{\partial x}-\tan\beta\right) \tag{22}$$

In order to obtain dimensionless expression for the discharge rate, use the variables of equation (4) in (22) and then define a dimensionless flow rate as

$$Q(X,\tau)=\frac{q(x,t)}{KL} \tag{23}$$

One obtains

$$Q(X,\tau)=-\left\{H(X,\tau)+\frac{h_0}{L}\right\}\left\{\left(\frac{\partial H}{\partial X}\right)-\tan\beta\right\}\cos^2\beta \tag{24}$$

Flow rate at the left and right interfaces of the aquifer can now be obtained by setting $X = 0$ and $X = 1$ respectively in equation (24). If these flow rates are denoted by Q_L and Q_R respectively, then

$$Q_L=-\left\{H(X=0,\tau)+\frac{h_0}{L}\right\}\left\{\left(\frac{\partial H}{\partial X}\right)_{X=0}-\tan\beta\right\}\cos^2\beta \tag{25a}$$

and

$$Q_R=-\left\{H(X=1,\tau)+\frac{h_0}{L}\right\}\left\{\left(\frac{\partial H}{\partial X}\right)_{X=1}-\tan\beta\right\}\cos^2\beta \tag{25b}$$

Values of $H(X = 0, \tau)$ and $H(X = 1, \tau)$ can be readily obtained from equations (6b) and (6c). Furthermore, partial derivatives involved herein are derived from equation (14) as follows:

$$\left(\frac{\partial H}{\partial X}\right)_{X=0} = h'_L\left\{\left(\alpha - \sqrt{\alpha^2-\lambda'_1}\coth\sqrt{\alpha^2-\lambda'_1}\right)\left(1-e^{-\lambda'_1\tau}\right) - \sum_{n=1}^{\infty}\frac{2n^2\pi^2}{(\alpha^2+n^2\pi^2)\left(\frac{\alpha^2+n^2\pi^2}{\lambda'_1}-1\right)}\left(1-e^{-(\alpha^2+n^2\pi^2)\tau}\right)\right\} + e^{-\alpha}h'_R\left\{\sqrt{\alpha^2-\lambda'_2}\left(1-e^{-\lambda'_2\tau}\right)\operatorname{csch}\sqrt{\alpha^2-\lambda'_2} + \sum_{n=1}^{\infty}\frac{2n^2\pi^2(-1)^n}{(\alpha^2+n^2\pi^2)\left(\frac{\alpha^2+n^2\pi^2}{\lambda'_2}-1\right)}\left(1-e^{-(\alpha^2+n^2\pi^2)\tau}\right)\right\} + N'\sum_{n=1}^{\infty}\frac{2n^2\pi^2\{1-(-1)^n e^{-\alpha}\}}{(\alpha^2+n^2\pi^2)^2}\left(1-e^{-(\alpha^2+n^2\pi^2)\tau}\right) \quad (26a)$$

and

$$\left(\frac{\partial H}{\partial X}\right)_{X=1} = -e^{\alpha}h'_L\left\{\sqrt{\alpha^2-\lambda'_1}\left(1-e^{-\lambda'_1\tau}\right)\operatorname{csch}\sqrt{\alpha^2-\lambda'_1} + \sum_{n=1}^{\infty}\frac{2n^2\pi^2(-1)^n}{(\alpha^2+n^2\pi^2)\left(\frac{\alpha^2+n^2\pi^2}{\lambda'_1}-1\right)}\left(1-e^{-(\alpha^2+n^2\pi^2)\tau}\right)\right\} + h'_R\left\{\left(\alpha + \sqrt{\alpha^2-\lambda'_2}\coth\sqrt{\alpha^2-\lambda'_2}\right)\left(1-e^{-\lambda'_2\tau}\right) + \sum_{n=1}^{\infty}\frac{2n^2\pi^2}{(\alpha^2+n^2\pi^2)\left(\frac{\alpha^2+n^2\pi^2}{\lambda'_2}-1\right)}\left(1-e^{-(\alpha^2+n^2\pi^2)\tau}\right)\right\} - N'\sum_{n=1}^{\infty}\frac{2n^2\pi^2\{1-(-1)^n e^{\alpha}\}}{(\alpha^2+n^2\pi^2)^2}\left(1-e^{-(\alpha^2+n^2\pi^2)\tau}\right) \quad (26b)$$

Initial flow rate at the ends $X = 0$ and $X = 1$ can be obtained by setting $\tau = 0$ in equations (25a) and (25b). The corresponding expressions for $(\partial H/\partial X)_{X=0}$ and $(\partial H/\partial X)_{X=1}$ vanish uniformly at $t = 0$. This yields in dimensional form

$$q_L = q_R = kh_0 \sin\beta\cos\beta \quad (27)$$

On the other hand, the steady-state value of flow rates at $X = 0$ and $X = 1$ can be deduced in a similar manner by setting $\tau \to \infty$ in (24a) and (24b). Noting that $H(X = 0, \tau \to \infty) = h_L'$, $H(X = 1, \tau \to \infty) = h_R'$, the final expressions for dimensionless form of steady stare flow rate are

$$q_L^{\infty} = -Kh_L\cos^2\beta\left\{\alpha(h'_R-h'_L)e^{-\alpha}\operatorname{csch}\alpha + N'\sum_{n=1}^{\infty}\frac{2n^2\pi^2\{1-(-1)^n e^{-\alpha}\}}{(\alpha^2+n^2\pi^2)^2} - \tan\beta\right\} \quad (28)$$

$$q_R^{\infty} = -Kh_R\cos^2\beta\left\{\alpha(h'_R-h'_L)e^{\alpha}\operatorname{csch}\alpha - N'\sum_{n=1}^{\infty}\frac{2n^2\pi^2\{1-(-1)^n e^{\alpha}\}}{(\alpha^2+n^2\pi^2)^2} - \tan\beta\right\} \quad (29)$$

5. NUMERICAL SOLUTION OF THE NONLINEAR BOUSSINESQ EQUATION

Analytical solutions developed in preceding sections are based on linearization of the Boussinesq equation. In order to assess the impact of linearization on the results obtained, the nonlinear Boussinesq equation (1) is solved using Du Fort and Frankel scheme. This scheme is an explicit and unconditionally stable scheme based on finite difference scheme discretization of parabolic nonlinear partial differential equation. Rewrite equation (1) as

$$\frac{\partial h}{\partial t} = A\left[\left(\frac{\partial h}{\partial x}\right)^2 - \tan\beta\,\frac{\partial h}{\partial x} + h\frac{\partial^2 h}{\partial x^2}\right] + \frac{N}{S} \quad (30)$$

where $A = S/(K\cos^2\beta)$. Use central difference for time as well as space derivatives and simplify it further to get

$$h_m^{n+1}\left(\frac{1}{2\Delta t}+A\frac{h_m^n}{(\Delta x)^2}\right) = A\left\{\left(\frac{h_{m+1}^n-h_{m-1}^n}{2\Delta x}\right)^2-\tan\beta\left(\frac{h_{m+1}^n-h_{m-1}^n}{2\Delta x}\right)+h_m^n\left(\frac{h_{m+1}^n-h_m^{n-1}+h_{m-1}^n}{(\Delta x)^2}\right)\right\} + \frac{h_m^{n-1}}{2\Delta t}+\frac{N}{S} \quad (31)$$

where $n \geq 2$. In order to initiate the scheme, one would require the value of $h_m{}^2$ which can be obtained by using forward difference discretization of equation (30) and setting n = 1 in it, yielding

$$h_m^2\left(\frac{1}{\Delta t}\right)=A\left\{\left(\frac{h_{m+1}^1-h_{m-1}^1}{2\Delta x}\right)^2-\tan\beta\left(\frac{h_{m+1}^1-h_{m-1}^1}{2\Delta x}\right)+h_m^1\left(\frac{h_{m+1}^1-2h_m^1+h_{m-1}^1}{(\Delta x)^2}\right)\right\}+\frac{h_m^1}{\Delta t}+\frac{N}{S} \quad (32)$$

The initial and boundary conditions are discretized as

$$h_m^1 = h_0 \quad (33a)$$

$$h_1^n = h_L-(h_L-h_0)e^{-\lambda_1 n} \quad (33b)$$

$$h_{M+1}^n(x=L,t>0)=h_R-(h_R-h_0)e^{-\lambda_2 n} \quad (33c)$$

where M Δx = L.

6. DISCUSSION OF RESULTS

The results obtained in this study are demonstrated with a numerical example with values of aquifer parameter as given in Table 1.

Table 1: Numerical values of aquifer parameters used in the illustrative example

Parameter	K	S	β	L	h_0	h_L	h_R	N	λ_1	λ_2
Value	2.5 m/d	0.2	3 deg	100 m	7 m	10 m	12 m	6 mm/h	0.5 d^{-1}	0.7 d^{-1}

Analytical expression as given in equation (14) is used for determination of water head height h (x, t) in the entire aquifer domain at multiple instants of time. Furthermore, Du Fort and Frankel scheme as described in Section 5 is used for calculating h (x, t) at the same instants. The aim is not only to obtain water head distribution in the aquifer, but also compare analytical and numerical solutions to assess validity of linearization used in this study. Numerical experiments were also carried out to assess convergence of infinite series present in the right-hand side of equations (14) and (26). It is observed that the infinite series are fast converging and first 50 terms more than enough to characterize the whole series. Water head heights obtained from analytical and numerical solutions are plotted in Figure 2. Here, the continuous and dotted curves are analytical and numerical solution respectively. Values of water head height from analytical and numerical solutions are also listed in Table 2. Both solutions are almost coincidental at initial instants of time (t = 1, 2 and 5 days). Even for large value of time such as t = 50 d, the difference between analytical and numerical solutions remains in the range 0 to 1.87%. Excellent agreement between analytical and numerical solutions establishes the validity of linearization of Boussinesq equation based on Marino (1973) technique. Water table evolves as a mound whose peak is slightly drifted towards the right end of the aquifer. The water head profiles stabilize to a steady state value for t ≥ 50 d.

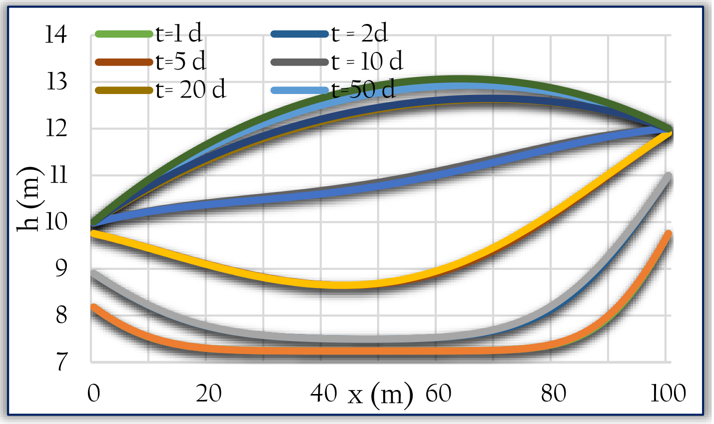

Figure 2: Dynamic profiles of water table in 3 deg sloping aquifer under constant recharge

Table 2: Comparison of water head height obtained from analytical and numerical solution

x	t = 5d		t =10d		t = 20d		t = 50d	
	h_a	h_n	h_a	h_n	h_a	h_n	h_a	h_n
10	9.43199	9.427381	10.24343	10.2289	10.7435	10.7821	10.90006	10.94318
20	9.08943	9.081737	10.41123	10.37562	11.33779	11.39192	11.59431	11.68512
30	8.81137	8.802517	10.54072	10.48452	11.80864	11.86291	12.12448	12.25437
40	8.66014	8.656667	10.6787	10.60729	12.16992	12.21588	12.51392	12.66742
50	8.68104	8.696944	10.85914	10.77813	12.42798	12.46251	12.7741	12.9342
60	8.91941	8.965423	11.09346	11.00942	12.58359	12.60819	12.90884	13.05794
70	9.41663	9.480158	11.36681	11.28752	12.63252	12.64864	12.91533	13.03746
80	10.15642	10.20802	11.64238	11.57877	12.56502	12.57396	12.7823	12.86429
90	11.03654	11.05889	11.87097	11.83484	12.49105	12.36665	12.49105	12.52573

For a horizontal aquifer under constant recharge, the heights of water table were calculated using both analytical and numerical solutions. When steady-state is achieved, it is observed that the peak of the mound occurs near x = 60.5 m. In order to get a fair idea of the mound height and location of its peak, an iterative scheme based on equation (19) is developed wherein the average saturated depth involved in N ′ is replaced by $(h_0 + h_a)/2$, where h_a is the height of groundwater mound at the steady-state. The dimensional form of equation (19) becomes

$$h_a = h_R + \frac{c}{8(h_0 + h_a)}\{1 - 2(h_R - h_L)(h_a + h_0)\}^2 \tag{34}$$

where $c = 2KL^2N$. Now, define an iterative method

$$h_a^{i+1} = h_R + \frac{c}{8(h_0 + h_a^i)}\{1 - 2(h_R - h_L)(h_a + h_a^i)\}^2 \tag{35}$$

where i = 0, 1, 2... Equation (29) can be used iteratively with $h_a^0 = h_0$. The scheme is iterated till two consecutive values of h_a^{i+1} become same. The resulting value h_a is then used in expression of X_a to approximate the spatial location of the peak of mound. When the scheme was implemented with aforementioned data, the mound height obtained is 13.45334 m and occurred at x_a = 60.65 m. The ease with which mound height can be calculated using equation (35) is one of the major contributions of this study.

Water table fluctuations in the absence of vertical recharge are plotted in Figure 3. Steady state profile is dependent on the slope of the aquifer's base and is given by

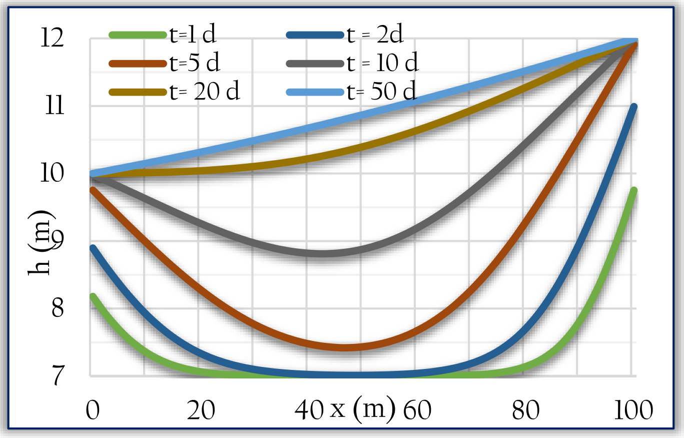

Figure 3: Dynamic profiles of water table in 3 deg sloping aquifer without recharge

$$H(X) = \frac{(h_L' e^{\alpha} - h_R' e^{-\alpha}) + e^{-\alpha(1-2X)}(h_R' - h_L')}{2 \sinh \alpha} \tag{36}$$

To get a better insight in how recharge rate influences flow mechanism in the aquifer, water head height h at the middle point of the aquifer, i.e. x = 50 m is plotted against time t for various values of recharge rate N. The plots reveal that the water head height increases with recharge rate. Furthermore, the steady-state value at which water table eventually settles also increase with recharge rate.

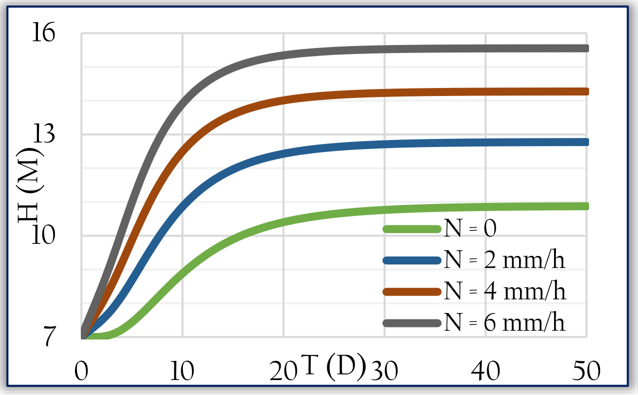

Figure 4: Water head height at the midpoint of the aquifer for various values of N

Stream rise rate parameter λ plays a deterministic role in the development of transient profiles of water head, but has no say in the steady state values of water head. In the following Figure 5, water head heights at the midpoint of the aquifer are plotted for various values of λ_1 and λ_2. For ease of handling the results, we have considered $\lambda_1 = \lambda_2$. It is shown in the figure that when the stream-stage rise is fast, the water table in the aquifer grows rapidly. However, the steady-state values are independent of the stream rise rate.

Flow rate hydrographs based on equation (25b) are plotted for various values of recharge rate λ_2. During initial stages, the outflow at the right end X = 1 decreases. It then attains a minimum value which might be positive (out flow) or negative (inflow) depending on the parameter λ_2 and then increases. A fast change in stream-stage at the right end leads to inflow in the aquifer. The flow rate eventually attains a steady-state value given by equation (29).

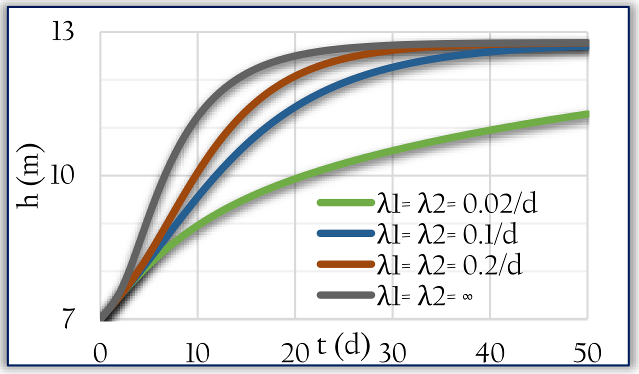

Figure 5: Water head height at the midpoint of the aquifer for various values of λ_1 and λ_2

Discharge rate q(x = L, t) is influenced by the downward percolation rate N. In the following Figure 7, the flow rate is plotted against time for various values of N. It is observed that transient as well as steady-state values increase with N. The vertical seepage is partly used in groundwater mound and remaining water flows out through the end x = L.

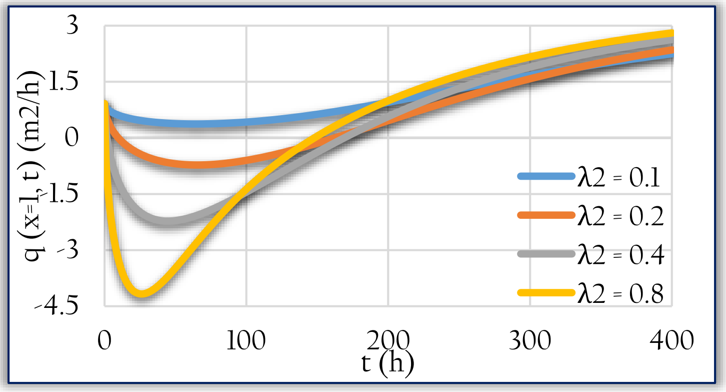

Figure 6: Flow rate hydrograph at stream-aquifer right end X = 1 for various values of λ_2

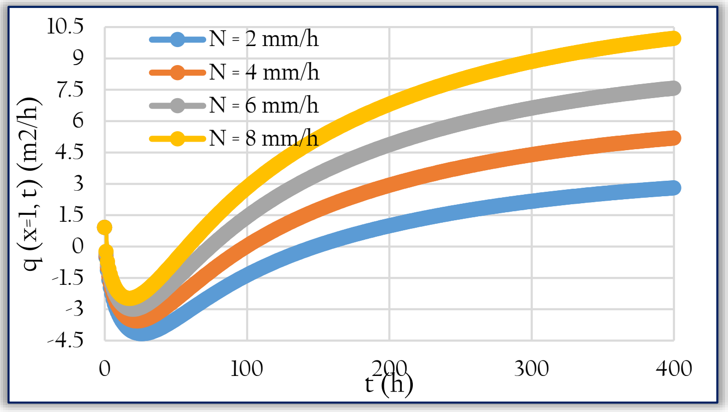

Figure 7: Flow rate hydrograph at stream-aquifer right end X = 1 for various values of N

7. CONCLUSIONS

New analytical solutions are developed to estimate water table fluctuations and discharge rate in a archetypical ditch-drain aquifer system underlain by a sloping impervious bed. The solutions are obtained from Boussinesq equation, the validity of whose linearization has been critically assessed by a numerical scheme. The solutions developed here have the capability of predicting spatio-temporal variations in water table and discharge rate at either ends of the aquifer. An iterative scheme for determination of mound height is derived from main results and illustrated with numerical examples. The results developed here can be used as test cases for numerical modelling and guideline for experimental work.

References:

[1] Akylas E, Koussis AD (2007) Response of sloping unconfined aquifer to stage changes in adjacent stream. I. Theoretical analysis and derivation of system response functions. J Hydrol 338(1-2): 85-95.
[2] Bansal RK, Das SK (2009) An analytical study of water table fluctuations in unconfined aquifers due to varying bed slopes and spatial location of the recharge basin. J Hydrol Eng 15(11): 909–917
[3] Bansal RK, Das SK (2010) An analytical study of water table fluctuations in unconfined aquifers due to varying bed slopes and spatial location of the recharge basin. J Hydrol Eng 15(11): 909–917
[4] Bansal RK, Das SK (2011) Response of an unconfined sloping aquifer to constant recharge and seepage from the stream of varying water level. Water Resour Manag 25: 893-911.
[5] Bansal RK (2012) Groundwater fluctuations in sloping aquifers induced by time-varying replenishment and seepage from a uniformly rising stream. Trans in Porous Media 92(2): 817-836, 2012
[6] Bansal RK (2013) Groundwater flow in sloping aquifer under localized transient recharge: Analytical study. J Hydraul Eng 139(11): 1165-1174.
[7] Bansal RK, Teloglou IS (2013) An analytical study of groundwater fluctuations in unconfined leaky aquifers induced by multiple localized recharge and withdrawal. Global NEST J 15(3): 394-407
[8] Baumann P (1952) Groundwater movement controlled through spreading. Amer Soc Civ Eng Trans 117: 1024–1074.
[9] Boyraz U, Kazezyilmaz-Alhan CM (2018) Solutions for groundwater flow with sloping stream boundary: analytical, numerical and experimental models. Hydrol Res 49(4): 1120-1130.
[10] Chang C-H, Huang C-S, Yeh H-D (2017) Analysis of three dimensional unsaturated- saturated flows induced by localized recharge in unconfined aquifers. Hydrol Earth Sys Sci 22:3951-3963
[11] Chang YC, Yeh HD (2007) Analytical solutions for groundwater flow in an anisotropic sloping aquifer with arbitrary located multiwells. J Hydrol 347(1-2): 143–152
[12] Chapman TG (1980) Modeling groundwater flow over sloping beds. Water Resour Res 16(6): 1114–1118.
[13] Childs EC (1971) Drainage of groundwater resting on sloping beds. Water Resour Res 7: 1256–1263.
[14] Cooper-Jr. HH, Rorabaugh MI (1963) Ground-water movements and bank storage due to flood stages in surface streams. USGC Water Supply Paper 1536-J: 343-366.
[15] Dagen G (1967) Linearized solution of free surface groundwater flow with uniform recharge, J Geophys Res 72:1183-1193.
[16] Dicker D, Sevian WA (1965) Transient flow through porous medium. J Geophysical Res 70(20): 5043-5054.

[17] Hantush MS (1967) Growth and decay of groundwater mounds in response to uniform percolation. Water Resour Res 3(1): 227–234
[18] Hogarth WL, Li L, Lockington DA, Stagnitti F, Parlange MB, Barry DA, Parlange JY (2014) Analytical approximation for the recession of a sloping aquifer. Water Resou Res 50(11): 8564-8570.
[19] Hornerberger GM, Ebert J, Remson I (1970) Numerical solution of the Boussinesq equation for aquifer-stream interaction. Water Resour Res 6(2): 601-608.
[20] Ireson AM, Butler AP (2013) A critical assessment of simple recharge models: application to the UK Chalk. Hydrol Earth Syst Sci 17: 2083-2096.
[21] Lin C (1972): Digital simulation of the Boussinesq equation for a water table aquifer. Water Resour Res 8(3): 691-698.
[22] Marino MA (1973) Water table fluctuation in semi pervious stream unconfined aquifer systems. J Hydrol 19(1): 43-52.
[23] Manglik A, Rai SN, Singh RN (1997) Response of an unconfined aquifer induced by time varying recharge from a rectangular basin. Water Resour Manage 11:185-196
[24] Polubarinova-Kochina PY (1962) Theory of Groundwater Movement. Princeton University Press, Princeton, N. J.
[25] Rai SN, Singh RN (1995) Two dimensional modeling of water table fluctuation in response to localized transient recharge. J Hydrol 167: 167-174.
[26] Rai SN, Singh RN (1996) On the prediction of groundwater mound formation due to transient recharge from rectangular area. Water Resour Manage 10(3): 189-196.
[27] Ram S, Chauhan HS (1987) Analytical and experimental solutions for drainage of sloping lands with time-varying recharge. Water Resou Res 23(6): 1090-1096
[28] Rao NH, Sharma PBS (1981) Recharge from rectangular areas to finite aquifers. J Hydrol 53(3-4): 269-275.
[29] Stagnitti F, Li L, Parlange JY, Brutsaert W, Lockington DA, Steenhuis TS, Hogarth WL (2004) Drying front in a sloping aquifer: Nonlinear effects. Water Resou Res 40(4)
[30] Teloglou IS, Bansal RK (2012) Transient solution for stream–unconfined aquifer interaction due to time varying stream head and in the presence of leakage. J Hydrol 428–429: 68–79
[31] Teloglou IS, Zissis ST, Pangopoulos CA (2008) Water table fluctuation in aquifers overlying a semi-impervious layer due to transient recharge from a circular basin. J Hydrol 348: 215-223.
[32] Troch P, Van Loon E, Hilberts A (2002) Analytical solutions to a hillslope-storage kinematic wave equation for subsurface flow. Adv in Water Resou 25(6): 637-649.
[33] Troch PA, Berne A, Bogaart P, Harman C, Hilberts AG, Lyon SW, Teuling AJ (2013) The importance of hydraulic groundwater theory in catchment hydrology: The legacy of Wilfried Brutsaert and Jean-Yves Parlange. Water Resou Res 49(9): 5099-5116
[34] van der Spek JE, Boggard TA, Bakker M (2013) Characterization of groundwater dynamics in landslides in varved clays. Hydrol Earth Syst Sci 17: 2171-2183.
[35] Verhoest NE, Troch PA (2000) Some analytical solutions of the linearized Boussinesq equation with recharge for a sloping aquifer. Water Resou Res 36(3): 793-800
[36] Yeh HD, Chang YC (2013) Recent advances in modelling of well hydraulics. Adv Water Resour 51: 27-51
[37] Yeh WWG (1970) Nonsteady flow to surface reservoir. J Hydraul Div ASCE 96(3): 609-618.
[38] Zomorodi K (1991) Evaluation of the response of a water table to a variable recharge. Hydrol Sci J 36(1): 67–78

ISSN 1584 – 2665 (printed version); ISSN 2601 – 2332 (online); ISSN-L 1584 – 2665