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POST-BUCKLING ANALYSIS OF AN AXIALLY LOADED PRISMATIC COLUMN

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Abstract: The need to improve on analysis and design of structural members for overall weight, material and financial economy is indispensable. This paper presents the results of post-buckling analysis of an axially loaded prismatic column. The theory of elasticity was employed to formulate the total potential energy functional for a prismatic column in the post buckling regime. The derived functional was minimized with respect to deflection function to obtain the governing equation of equilibrium. The governing equation was then solved to obtain the deflection equation. Again, the functional was minimized with respect to the coefficient of the obtained deflection function to obtain the formula for calculating the post-buckling load of the prismatic column. A numerical example was given to demonstrate the applicability of the present formulation. The prismatic column was analyzed within the post buckling regime to determine the stiffness coefficients. It was shown that as the deflection-thickness (Δ/t) ratio increased, the combined buckling stress increased geometrically. It was found that the critical buckling load obtained from the present method and those obtained from literature showed consistency. It was also found that at the buckled state of the prismatic column, the yield strength of the column material has not been attained and deflection is very small. This gives room for the column to take on more loads before it fails either by excessive deflection or by stress exceeding the material strength.
Keywords: post-buckling regime, prismatic column, elasticity, deflection equation, stiffness coefficients, yield strength

1. INTRODUCTION

Column is a structural member whose failure is of dare consequence to a structural assemblage. Buckling has been a structural challenge, and is a form of structural instability problem. Stability of structures can be explored by calculating its critical force. This means the force (load) analogous to a state in which discomposure of the buckle state does not disrupt the equilibrium between the internal and external forces (Stipica et al., 2012). After the works of Euler several years ago in solving the first instability problem and proposing values of critical loads for different end conditions, many scholars have carried out research work on this area. Jifeng et al. (2013) found that the main reason of buckling analysis is to determine the critical load and buckling deformation. This would enable good understanding of the buckling resistance and the load carrying capability of the member. As the member is gradually being subjected to incremental load, a second cycle of deformation with equilibrium path becoming quite shallower is noticed, resulting in limited load-carrying capacity of such column (Loannidis, 1994). Jifenget al. (2013) found that the post-buckling characters of structural members generally represent extremely large deflections and need non-linear approach for solution. Kounadis et al. (2006) found that critical state of perfect columns results in stable symmetric bifurcation point using a technique of an approximate analytic method. Though other researchers have examined the post buckling regime with discrete approache such as the differential quadrature method, Zhangxian and Xinwei (2011) employed a different approach and found that when loads of high magnitudes are applied, an assumption such as small rotation in obtaining several scales results is no longer valid.
Solving post buckling problems involves a good knowledge of the support conditions in order to establish the characteristics equation which is employed in solving the critical loads (Phungpaingam and Chucheepsakul, 2005). The works of Mihael (2011) employed approximate formulas for non-linearly elastic columns which were made from Ludwick materials in the post-buckling regime for support conditions free-clamp, both hinged and both clamp. There was good agreement between the analytical and numerical solutions.
This paper examines the post-buckling analysis of an axially loaded prismatic column, using the theory of elasticity in a different approach at varying support conditions to fully utilize the yield strength of column.

2. FORMULATION OF TOTAL POTENTIAL ENERGY FUNCTIONAL

A constrained structural system (a structure that is restrained at some portions) will deform when forces are applied on it. The Potential energy (PE) of a structural system is defined as the sum of the strain energy (U) and the external work (V) (That is: PE=U+V)

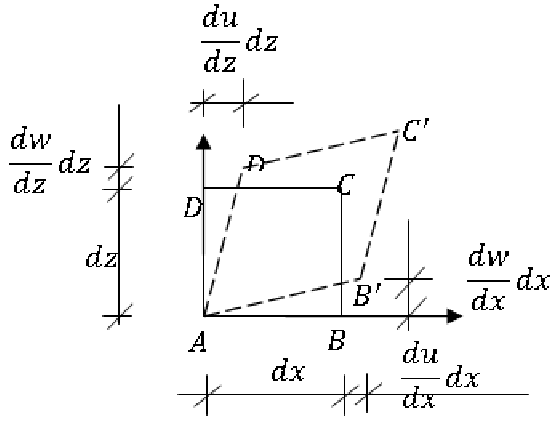

Figure 1: Large deformation of a finite element of beam

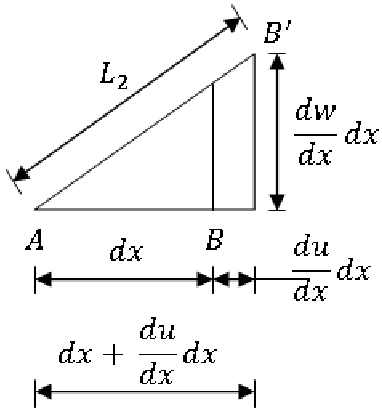

Figure 2: Right angled triangle formed after point B displacement

The displacements are allowed along the x and z coordinates. They are axial and normal (lateral) displacements designated as u and w respectively. The normal displacement will be referred to as deflection. The displacements of a finite element in a beam under large deflection are shown in Figure 1 above.

After large deformation of a beam, point B in the finite element shown on Figure 1 moves to new point, B'. In the same manner, point D moves to point D'. These movements create new shape as shown in Figure 2.

Applying Pythagoras theorem (Figure 2) yields:

$$L_2^2 = \left[dx + \frac{du}{dx}dx\right]^2 + \left[\frac{dw}{dx}dx\right]^2 \quad (1)$$

Expanding equation (1) and neglecting small quantity and applying Binomial theorem yields:

$$L_2 = dx + \frac{du}{dx}dx + \frac{1}{2}\left(\frac{dw}{dx}\right)^2 dx \quad (2)$$

However, the original length of line AB is dx. Then, the difference in length after large deflection is given as:

$$dL = L_2 - dx \quad (3)$$

Substituting equation (2) into equation (3) gives:

$$dL = \frac{du}{dx}dx + \frac{1}{2}\left(\frac{dw}{dx}\right)^2 dx \quad (4)$$

From Kirchhoff's assumption of zero vertical shear deformation for thin beam, it is obtained that:

$$u = u_0 - z\frac{dw}{dx} \quad (5)$$

where u0 = axial displacement of the middle surface of the beam.

Substituting equation (5) into equation (4) gives:

$$dL = dx\left[\frac{du_0}{dx} - z\frac{d^2w}{dx^2} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2\right] \quad (6)$$

The normal axial strain, defined as the ratio of increase in length to original length is given as:

$$\varepsilon = \frac{dL}{dx} \quad (7)$$

Substituting equation 6 into equation 7 gives:

$$\varepsilon = \left[\frac{du_0}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2\right] + \left[-z\frac{d^2w}{dx^2}\right] \quad (8)$$

From equation (8), it can be inferred that the strain is made up of membrane strain, εm and flexural (bending) strain, εb given respectively as:

$$\varepsilon_m = \frac{du_0}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2 \quad (9)$$

$$\varepsilon_b = -z\frac{d^2w}{dx^2} \quad (10)$$

From Hooke's law, it is obtained that the membrane and bending stresses are:

$$\sigma_m = E\,\varepsilon_m = E\left[\frac{du_0}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2\right] \quad (11)$$

$$\sigma_b = E\,\varepsilon_B = -zE\frac{d^2w}{dx^2} \quad (12)$$

3. POST BUCKLING REGIME TOTAL POTENTIAL ENERGY FUNCTIONAL

Strain energy is the product of strain, stress and volume of a matter. The total average strain energy for the entire beam is obtained through indefinite summation of the elemental strain energy as:

$$U = \frac{b}{2}\int\int_{\frac{-t}{2}}^{\frac{t}{2}}(\sigma\varepsilon)dxdz \tag{13}$$

As there exists membrane and bending strains and stresses, so also there exists membrane and bending strain energies. That is:

$$U_m = \frac{b}{2}\int\int_{\frac{-t}{2}}^{\frac{t}{2}}(\sigma_m\varepsilon_m)dxdz \tag{14}$$

$$U_b = \frac{b}{2}\int\int_{\frac{-t}{2}}^{\frac{t}{2}}(\sigma_b\varepsilon_b)dxdz \tag{15}$$

Substituting equations (9) and (11) and equations (10) and (12) into equations (14) and (15) respectively gives:

$$U_m = \frac{EA}{2}\int\left[\left(\frac{du_0}{dx}\right)^2 + \frac{du_0}{dx}\left(\frac{dw}{dx}\right)^2 + \frac{1}{4}\left(\frac{dw}{dx}\right)^4\right]dx \tag{16}$$

$$U_b = \frac{EI}{2}\int\left(\frac{d^2w}{dx^2}\right)^2 dx \tag{17}$$

where $A = bt$ and $I = \frac{bt^3}{12}$

After buckling of the slender column, two stresses act on it. The stresses are axial and bending stresses defined as:

$$\sigma_a = \frac{N}{bt} \tag{18}$$

$$\sigma_b = \pm\frac{6N\Delta}{bt^2} \tag{19}$$

Where positive and negative signs stand for compression and tensile bending stresses respectively, Δ is the maximum value of deflection of the column. Note that Δ is zero just before buckling. Multiplying equations (18) and positive equation (19) by curvature $(\frac{d^2w}{dx^2})$ and integrating the outcomes with respect to deflection, and adding the two results give the external work of the column in post buckling regime:

$$V = \frac{N}{2}\int\left[1 - 4\frac{\Delta}{t}\right]\left(\frac{dw}{dx}\right)^2 dx \tag{20}$$

Total potential energy is the summation of strain energy and external work given as:

$$\Pi = U_b + U_m - V \tag{21}$$

Substituting equations (16), (17) and (20) into equation (21) gives:

$$\Pi = \frac{EI}{2}\int\left(\frac{d^2w}{dx^2}\right)^2 dx + \frac{EA}{2}\int\left[\left(\frac{du_0}{dx}\right)^2 + \frac{du_0}{dx}\left(\frac{dw}{dx}\right)^2 + \frac{1}{4}\left(\frac{dw}{dx}\right)^4\right]dx - \frac{N}{2}\left[1 - 4\frac{\Delta}{t}\right]\int\left(\frac{dw}{dx}\right)^2 dx \tag{22}$$

4. GOVERNING AND COMPATIBILITY EQUATIONS AND DISPLACEMENT FUNCTIONS

Minimizing the total potential energy functional with respect to deflection gives the governing equation as:

$$\frac{d\Pi}{dw} = EI\int\frac{d^4w}{dx^4}dx + EA\int\frac{d^2w}{dx^2}\left[\frac{du_0}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2\right]dx - N\left(1 - 4\frac{\Delta}{t}\right)\int\frac{d^2w}{dx^2}dx = 0 \tag{23}$$

Minimizing the total potential energy functional with respect to axial displacement gives the compatibility equation as:

$$\frac{d\Pi}{du_0} = EA\int\frac{d}{dx}\left[\frac{du_0}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2\right]dx = 0 \tag{24}$$

Solving equation (24) gives:

$$\frac{du_0}{dx} = -\frac{1}{2}\left(\frac{dw}{dx}\right)^2 \tag{25}$$

Substituting equation (25) into equation (23) gives:

$$\int\left[\frac{d^4w}{dx^4} - \frac{N}{EI}\left(1 - 4\frac{\Delta}{t}\right)\frac{d^2w}{dx^2}\right]dx = 0 \tag{26}$$

The ready solution of equation (26) in terms of non-dimensional coordinate is:

$$w = c_0 + c_1R + c_2d_1e^{ikR} + c_2d_2e^{-ikR} \tag{27}$$

where:

$$k^2 = -\frac{NL^2}{EI}\left(1 - 4\frac{\Delta}{t}\right) \tag{28}$$

The polynomial and trigonometric transformations of equation (27) are respectively:

$$w = \sum_{i=0}^{i=\infty} a_i R^i = Ah \quad (29)$$

$$w = a_0 + a_1 R + a_2 \cos BR + a_3 \sin BR = ah \quad (30)$$

Truncating equation (29) at a term whose power of R is four gives the approximate deflection function in polynomial form as:

$$w = a_0 + a_1 R + a_2 R^2 + a_3 R^3 + a_4 R^4 \ = ah \quad (31)$$

Satisfying the boundary conditions of the columns, peculiar deflection functions (both in polynomial form and trigonometric form) are obtained as presented in Table 1.

5. FORMULA FOR CALCULATING THE POST-BUCKLING REGIME LOAD

Minimization of the total potential energy functional with respect to deflection coefficient gives the formula for calculating the post-buckling load. Substituting equations (25), (30) and (31) into equation (22) and writing the outcome in non-dimensional coordinate terms gives:

$$\Pi = \frac{EIa^2}{2L^4}\int\left(\frac{d^2h}{dR^2}\right)^2 LdR - \frac{Na^2}{2L^2}\left(1 - 4\frac{\Delta}{t}\right)\int\left(\frac{dh}{dR}\right)^2 LdR \quad (32)$$

Minimizing equation (32) with respect to 'a' gives:

$$\frac{d\Pi}{da} = \frac{EIa}{L^4}\int\left(\frac{d^2h}{dR^2}\right)^2 LdR - \frac{Na}{L^2}\left(1 - 4\frac{\Delta}{t}\right)\int\left(\frac{dh}{dR}\right)^2 LdR = 0 \quad (33)$$

Rearranging equation (33) gives:

$$N = \left(\frac{1}{1 - 4\frac{\Delta}{t}}\right) \cdot \frac{EI}{L^2} \cdot \left(\frac{k_1}{k_2}\right) \quad (34)$$

where:

$$k_1 = \int_0^1 \left(\frac{d^2h}{dR^2}\right)^2 dR \quad (35)$$

$$k_2 = \int_0^1 \left(\frac{dh}{dR}\right)^2 dR \quad (36)$$

It has earlier been said that Δ is the maximum value of deflection. Hence, it is expressed as:

$$\Delta = w_{max} = a \,.\, h_{max} \quad (37)$$

Substituting equation (37) into equation (34) gives:

$$N = \left(\frac{1}{1 - 4\frac{\Delta}{t}}\right) \cdot \frac{EI}{L^2} \cdot \left(\frac{k_1}{k_2}\right) \quad (38)$$

The values of k1 and k3 for columns of various boundary conditions are as presented in Table 2. Rearranging equation (38) gives:

$$\frac{\Delta}{t} = \frac{1}{4} - \left(\frac{1}{4N}\right) \cdot \frac{EI}{L^2} \cdot \left(\frac{k_1}{k_2}\right) \quad (39)$$

Just before buckling, deflection is zero and buckling load is critical buckling load, Ncr. That is:

$$0 = \frac{1}{4} - \frac{1}{4N_{cr}} \cdot \frac{EI}{L^2} \cdot \left(\frac{k_1}{k_2}\right) \quad (40)$$

Re-arranging equation (40) gives:

$$N_{cr} = \cdot \frac{EI}{L^2} \cdot \left(\frac{k_1}{k_2}\right) \quad (41)$$

Substituting equation (41) into equation (38) gives:

$$N = \left(\frac{1}{1 - 4\frac{\Delta}{t}}\right) . N_{cr} \quad (42)$$

Substituting equation (41) into equation (39) gives:

$$\frac{\Delta}{t} = \frac{1}{4}\left[1 - \frac{N_{cr}}{N}\right] \quad (43)$$

Substituting equation (42) into equations (18) and (19) and summing their outcomes gives the combined stress in the column in the post-buckling regime:

$$\sigma_c = \frac{N_{cr}}{bt} \cdot \left(\frac{1 + 6\frac{\Delta}{t}}{1 - 4\frac{\Delta}{t}}\right) \quad (44)$$

For safety against mechanical yield of the column, the combined stress at all times must be less than the yield strength given as σy.

$$\sigma_y = \frac{N_y}{A} \tag{45}$$

An example problem: A prismatic column of length, 10m, cross section area, 50625mm2 and second moment of inertia, 213574219 mm4 is analyzed using the method developed herein. The young's elasticity modulus of the column material and yield strength of the material is 25kN/mm2 and 25 N/mm2 respectively. Determine also the critical length of this column, below which the critical buckling load is always more than the yield load.

6. RESULTS OF POST-BUCKLING ANALYSIS

The results of the stiffness coefficients corresponding to the various boundary conditions are presented in Table 2. Also, the results of the post-buckling analysis are presented in Table 3 and Table 4 respectively.

Table 1: Peculiar deflection functions for columns of various boundary conditions

Boundary conditions	Polynomial deflection functions, w (Present Method)	Trigonometric deflection functions, w (Phungpaingam and Chucheepsakul, 2005)
s-s	$w = a(R - 2R^3 + R^4)$	$w = a\,\mathrm{Sin}(\pi R)$
c-s	$w = a(1.5R^2 - 2.5R^3 + R^4)$	$w = a(\mathrm{Cos}\,2\pi R - 1)$
c-c	$w = a(R^2 - 2R^3 + R^4)$	$w = a(g_1 - g_1 R - g_1 \mathrm{Cos}\, g_1 R + \mathrm{Sin}\, g_1 R)$ $g_1 = 4.49340946$
c-f	$w = \frac{a}{6}\left(42\frac{15}{16}R^2 - 16\frac{5}{16}R^3 + R^4\right)$	$w = a\left(\mathrm{Cos}\frac{n\pi Q}{2} - 1\right)$

Table 2: k values for various boundary conditions

	Present Method			Phungpaingam and Chucheepsakul (2005)		
Line continuum	k_1	k_2	$N_{cr}\frac{L^2}{EI}$	k_1	k_2	$N_{cr}\frac{L^2}{EI}$
S – S	4.8	$\frac{17}{35}$	9.88	$\frac{m^4\pi^4}{2}$	$\frac{m^2\pi^2}{2}$	9.870
C - C	$\frac{4}{5}$	$\frac{2}{105}$	42	$8m^4\pi^4$	$2m^2\pi^2$	39.478
C - S	$\frac{9}{5}$	$\frac{3}{35}$	21	$\frac{{g_1}^6}{2}$ $g_1 = 4.49340946$	$\frac{{g_1}^4}{2}$	20.191
C - F	63.64505	25.2869288	2.517	$\frac{m^4\pi^4}{32}$	$\frac{m^2\pi^2}{8}$	2.467

Table 3: Combined buckling stress on column for various Δ/t ratios

Δ/t	0	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09	0.1
σ (MPa)	10.41	11.49	12.67	13.96	15.37	16.92	18.63	20.53	22.66	25.05	27.76
N (kN)	526.97	548.93	572.80	598.83	627.35	658.72	693.39	731.91	774.96	823.40	878.29

Table 4: Critical buckling stress on column for various Δ/t ratios

L (mm)	6452.3	6452.4	6452.5	6452.6	6452.7	6452.8	6452.9	6453
σcr(MPa)	25.003	25.002	25.002	25.001	25.000	24.999	24.998	24.998
Ncr(kN)	1265.784	1265.744	1265.705	1265.666	1265.627	1265.587	1265.548	1265.509

7. DISCUSSION OF RESULTS

The result of this example is presented on Table 3 and Table 4. It is seen from Table 3 that the column fails mechanically when the ratio of deflection to thickness is up to 0.09. Below this value, the column strength (25 N/mm2) is more than the combined buckling stress. From Table 3, it can be that as the deflection-thickness (Δ/t) ratio increases, the combined buckling stress increases geometrically. From Table 4, it is seen there that the critical length, below which the critical buckling load is always greater than the yield load is 6452.8mm. This implies that as long as the length of the column (the one described herein) is less than 6452.8mm, the column will fail mechanically before reaching the buckling state. What this means it that such column shall not buckle. It would have collapsed before attaining a stress close to critical buckling stress.

8. CONCLUSION

The results of post-buckling analysis of an axially loaded prismatic column based the total potential energy functional for a prismatic column in the post buckling regime has been presented. It was found that the critical buckling load obtained from the present method and those obtained from literature showed consistency. It was also shown that as the deflection-thickness (Δ/t) ratio increased, the combined buckling stress increased geometrically. It was also found that the critical length, below which the critical buckling load is always greater than the yield load is 6452.8mm, implying that as long as the length of the column is less than 6452.8mm, the column will fail mechanically before reaching the buckling state. It would have collapsed before attaining a stress close to critical buckling stress. It was also found that at the buckled state of the prismatic column, the yield strength of the column material has not been attained and deflection is very small.

This gives room for the column to take on more loads before it fails either by excessive deflection or by stress exceeding the material strength.

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