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USE OF REFINED BEAM THEORY FOR FREE AND FORCED VIBRATION ANALYSIS OF A DEEP PRISMATIC BEAM

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Abstract: This paper applied the method outlined in Refined Beam theory 2 (RBT2) for the analysis of a deep prismatic beam. RBT2 made assumptions of the displacement functions that are used in the beam analysis, this present study however, modified the works of RBT2 by providing a simple approach of obtaining the displacement function to be used for any boundary condition. To achieve this, the governing general differential equation, which is derived from the total potential energy of the beam, is integrated. Again, the total potential energy functional of the beam is minimized with respect to the displacements coefficients. This gives the formulas for calculating the displacement coefficients and natural frequency. The results of an illustrative example showed excellent agreement with results of other theories. Of the theories that considered the effects of shear deformation, a maximum percentage difference of -5.799% was obtained in the results of the centre deflection while a maximum of 3.0477% was obtained in the results of the resonating frequency. However, there was a higher percentage difference in the results obtained when compared with the thin beam theory. The results, however, showed that the percentage difference reduced as the span-depth ratio increased. This small value of percentage difference show that this present method is a reliable method for the analysis of deep prismatic beams of any given boundary condition and aspect ratio.

Keywords: free vibration, shear deformation, vertical shear strain, deflection, displacement, potential energy, natural frequency, span-depth ratio

1. INTRODUCTION

One of the most common structural elements is the beam. Several studies have been carried out on the analysis of beam-type structures and a number of theories have been developed. The oldest and most popular beam theories are the Classical Beam Theory (CBT). In the CBT, it is assumed that the vertical line, which is initially straight and perpendicular to the mid-plane before bending remain straight and perpendicular to the mid-plane after bending. As a result of this assumption, both transverse shear and transverse normal strain are ignored (Mesut and Turgat, 2007; Gawali et al., 2011; Wang et al., 2000, Ibearugbulem et al., 2015). This assumption is useful for thin beams (when the span-depth ratio is more than 20). But for deep beams (with span-depth ratio less than 20), the assumption no longer yields accurate results (Ibarugbulem et al., 2015).

Another popular beam theory is the Timoshenko beam also known as the First Order Shear Deformation Theory – FSDT (Viskas and Ajah, 2016; Sayyad, 2011). This theory sought to address the limitation of the CBT, which ignored shear deformation effects. FSDT assumes that straight lines perpendicular to the mid-plane before bending remain straight, but no longer perpendicular to the mid-plane after bending. (Wang et al., 2000). According to Viskas and Ajay (2016), this assumption implied a constant shear stress distribution across the beam thickness. This necessitated the need for introducing a shear correction factor.

To avoid the use of the correction factor, several higher order shear deformation theories, referred to as refined beam theories have been developed. These theories introduce a displacement that is a function of polynomials. Ibearugbulem et al. (2015) noted that an erroneous assumption ($u_c = -z\frac{dw}{dx}$) was used in all the refined beam theories. The error came from the assumption that the total displacement is composed of the classical axial displacement and the shear displacements. That is ($u = u_c + u_s$). And from the CBT, the axial displacement is used, giving that $u_c = -z\frac{dw}{dx}$. They opined that such assumption implied zero vertical shear strain for the classical part of the strain. That is, "$\gamma_{xz} = \frac{du}{dz} + \frac{dw}{dx} = 0$". Following that observation, Ibearugbulem et al. (2015) developed two new theories that fully implemented the assumption that engineering vertical strain is not fully zero. However, the work of Ibearugbulem et al. (2015) depended on assumption of the displacement functions used in the developed theories. This present study modifies the refined beam theory 2 (RBT2) as proposed by Ibearugbulem et al. (2015) by presenting a simple approach to obtain displacement equations for any given support condition that could be used in RBT2. The present approach is simple and straightforward.

2. MODEL DEVELOPMENT

The basic assumptions of RBT2 of thick line continuum of small deflection used in the model development include the following:

- The line continuum material is elastic, homogenous and isotropic.
- The line continuum is straight (not bent) before loading.
- The out of plane displacement (w) of the middle surface of the line continuum is less than one-third of plate thickness.

- The middle x-y plane of the line continuum shall not stretch.
- The x-y plane and y-z plane shear strains are very small when compared with x-z shear strain. Hence, they are assumed to be zero.
- The normal strains ε_y, ε_z acting normally on the x-z plane and x-y plane respectively are so small when compared with normal strain ε_x acting normally on the y-z plane and x-z shear strain. Hence, they are assumed to be zero. Another consequence of this is that the Poisson's ratios for x-y plane and y-z plane are zero.
- The maximum x-z plane shear stress (τ_{xzm}) distributed through the line continuum thickness is the product of nominal x-z plane shear stress (τ_{xz}) and shear stress shape factor, G(z). That is:

$$\tau_{xzm} = \tau_{xz}G(z) \text{ and } \tau_{xz} = \frac{E}{2(1+\mu)}\gamma_{xz}. \text{ That is: } \tau_{xzm} = \frac{EG(z)}{2(1+\mu)}\gamma_{xz}$$

- Like the third order shear deformation theory, the slope of RBT2 is split into classical part and shear deformation part.

The axial displacement is considered to be made up of the classical part and the shear deformation part. This addition is:

$$u = u_c + u_s \tag{1}$$

The vertical shear strain is obtained by adding the classical and shear deformation part as follows:

$$\theta = \theta_c + \theta_s = \gamma_{xzc} + \gamma_{xzs} \tag{2}$$

The classical vertical shear strain is:

$$\gamma_{xzc} = \frac{du_c}{dz} + \frac{dw}{dx} \tag{3}$$

The shear deformation vertical shear strain is:

$$\gamma_{xzs} = \theta_s \tag{4}$$

According to Ibearugbulem et al. (2015), the two terms at the left hand side of equation (3) are complementary. That is they are equal to each other. Hence,

$$\frac{du_c}{dz} = \frac{dw}{dx} \tag{5}$$

Integrating equation (5) gives:

$$u_c = z\frac{dw}{dx} \tag{6}$$

Slope is primarily defined as the first derivative of deflection with respect to axial coordinate.
Thus, classical and shear deformation axial displacements are given respectively as:

$$u_c = z\theta_c \tag{7}$$

$$u_s = z\theta_s \tag{8}$$

Substituting equations (6) and (8) into equations (1) yields:

$$u = z\frac{dw}{dx} + z\theta_s \tag{9}$$

From the assumptions, ε_y, ε_z, γ_{xy} and γ_{yz} are zero. Therefore, the strain components are as follows:

$$\varepsilon_x = \frac{du}{dx} = \frac{d}{dx}\left(z\frac{dw}{dx} + z\theta_s\right) = z\frac{d^2w}{dx^2} + z\frac{d\theta_s}{dx} = \varepsilon_x = z\left(\frac{d^2w}{dx^2} + \frac{d\theta_s}{dx}\right) \tag{10}$$

$$\gamma_{xz} = \gamma_{xzc} + \gamma_{xzs} = \left(\frac{du_c}{dz} + \frac{dw}{dx}\right) + \theta_s = \left(\frac{dw}{dx} + \frac{dw}{dx}\right) + \theta_s = 2\frac{dw}{dx} + \theta_s \tag{11}$$

— Constitutive Relations

The relationships between stresses and strains of the thick line continuum are:

$$\sigma_x = E\,.\,\varepsilon_x \tag{12}$$

$$\tau_{xz} = \frac{E}{2(1+\mu)}\,.\gamma_{xz} \tag{13}$$

— Total Potential Energy Functional for a Vibrating Deep Beam

Strain energy is defined mathematically as:

$$U = \frac{1}{2}\int_0^L\int_0^b\int_{-\frac{t}{2}}^{\frac{t}{2}}(\sigma_x\varepsilon_x + \tau_{xz}\gamma_{xz})\,(dxdydz) \tag{14}$$

For a beam subjected to dynamic loading, the external work is in two parts; one due to bending and the other due to vibration. The external work is defined as:

$$V = \frac{-m\lambda^2}{2}\int_0^l w^2\,dx - q\int_0^l w\,dx \tag{15}$$

The algebraic summation of the strain energy and the external work is the total potential energy functional given as:

$$\Pi = \frac{1}{2}\int_0^L\int_0^b\int_{-\frac{t}{2}}^{\frac{t}{2}}(\sigma_x\varepsilon_x + \tau_{xz}\gamma_{xz})\,(dxdydz)\ \frac{-m\lambda^2}{2}\int_0^l w^2\,dx - q\int_0^l w\,dx \tag{16}$$

Substituting equation (10), (11), (12), and (13), into (16) gives:

$$\Pi = \frac{EI}{2}\int_0^L\left(\left[\frac{d^2w}{dx^2}\right]^2 + 2\left[\frac{d^2w}{dx^2}\right]\frac{d\theta_s}{dx} + \left[\frac{d\theta_s}{dx}\right]^2 + \frac{6\alpha^2\left\{4\left[\frac{dw}{dx}\right]^2 + 4\left[\frac{dw}{dx}\right]\theta_s + [\theta_s]^2\right\}}{L^2\times\{1+\mu\}}\right)dx$$

$$\frac{-m\lambda^2}{2}\int_0^l w^2\,dx - q\int_0^l w\,dx \tag{17}$$

where: α is the span-depth ratio (L/t)

— General Variation of Total Potential Energy Equation

General variation is the minimization of the total potential energy functional with respect to displacement functions. The total potential energy functional contains two displacement functions (deflection and slope). Differentiating equation (17) with respect to w and θ_s will yield two simultaneous equations.

$$\frac{d\pi}{dw} = \frac{d\pi}{d\theta_s}\ 0 \tag{18}$$

That is:

$$\frac{d\pi}{dw} = \frac{EI}{2}\int_0^L\left(2\frac{d^4w}{dx^4} + 2\frac{d^3\theta_s}{dx^3} + \frac{6\alpha^2\left\{8\frac{d^2w}{dx^2} + 4\frac{d\theta_s}{dx}\right\}}{L^2\times\{1+\mu\}}\right)dx - m\lambda^2\int_0^l w\,dx - q\int_0^l 1\,dx\ = 0 \tag{19}$$

and

$$\frac{d\Pi}{d\theta_s} = \frac{EI}{2}\int_0^L\left(2\frac{d^3w}{dx^3} + 2\frac{d^2\theta_s}{dx^2} + \frac{6\alpha^2\left\{4\frac{dw}{dx} + 2\theta_s\right\}}{L^2\times\{1+\mu\}}\right)dx = 0 \tag{20}$$

For pure bending, frequency is zero. With this equation (19) becomes:

$$EI\int_0^L\left(\frac{d^4w}{dx^4} + \frac{d^3\theta_s}{dx^3} + \frac{6\alpha^2\left\{8\frac{d^2w}{dx^2} + 4\frac{d\theta_s}{dx}\right\}}{L^2\times\{1+\mu\}}\right)dx - q\int_0^l 1\,dx = 0 \tag{21}$$

For free vibration, uniform distributed load is zero. With this equation (19) becomes:

$$\frac{E\,I}{2}\int_0^L\left(2\frac{d^4w}{dx^4} + 2\frac{d^3\theta_s}{dx^3} + \frac{6\alpha^2\left\{8\frac{d^2w}{dx^2} + 4\frac{d\theta_s}{dx}\right\}}{L^2\times\{1+\mu\}}\right)dx - m\lambda^2\int_0^l w\,dx = 0 \tag{22}$$

Equations (21) and (22) can be written in terms of the non-dimensional coordinate, R. This gives:

$$\frac{E\,I}{L^4}\int_0^1\left(\frac{d^4w}{dR^4} + L\frac{d^3\theta_s}{dR^3} + \frac{6\alpha^2}{\{1+\mu\}}\left\{4\frac{d^2w}{dR^2} + 2L\frac{d\theta_s}{dR}\right\}\right)dR - q\int_0^1 1\,dR = 0 \tag{23}$$

$$\frac{E\,I}{L^4}\int_0^1\left(\frac{d^4w}{dR^4} + L\frac{d^3\theta_s}{dR^3} + \frac{6\alpha^2}{\{1+\mu\}}\left\{4\frac{d^2w}{dR^2} + 2L\frac{d\theta_s}{dR}\right\}\right)dR - m\lambda^2\int_0^1 w\,dR = 0 \tag{24}$$

where:

$$R = \frac{x}{L} \tag{25}$$

Solving equation (20) gives:

$$\frac{d^3w}{dx^3} + \frac{d^2\theta_s}{dx^2} + e\left\{2\frac{dw}{dx} + \theta_s\right\} = 0 \tag{26}$$

where:

$$e = \frac{6\alpha^2}{L^2\times\{1+\mu\}} \tag{27}$$

Rearranging equation (26) gives:

$$\left[\frac{d^2}{dx^2}+2e\right]\frac{dw}{dx}+\left[\frac{d^2}{dx^2}+e\right]\theta_s=0 \tag{28}$$

The non trivial conditions for equation (28) to be zero are:

$$e=-0.5\frac{d^2}{dx^2}\text{ and }e=-\frac{d^2}{dx^2} \tag{29}$$

It is made out of equation (29) that:

$$e=\frac{6\alpha^2}{L^2\times\{1+\mu\}}=c_1\frac{d^2}{dx^2} \tag{30}$$

Where c_1 is a yet to be determined constant. Similarly, a condition for equation (26) to be zero is:

$$\theta_s=c_2\frac{dw}{dx} \tag{31}$$

Where c_2 is another yet to be determined constant. Substituting equations (30) and (31) into equation (21) gives:

$$EI\int_0^L\left(\frac{d^4w}{dx^4}+\frac{d^3\left[c_2\frac{dw}{dx}\right]}{dx^3}+8c_1\frac{d^4w}{dx^4}+4c_1\frac{d^3\left[c_2\frac{dw}{dx}\right]}{dx^3}\right)dx-q\int_0^l 1\,dx=0$$

That is:

$$EI\int_0^L\frac{d^4w}{dx^4}(1+c_2+8c_18c_1+4c_1c_2)\,dx-q\int_0^l 1\,dx=0 \tag{32}$$

Rearranging equation (32) and writing the outcome in non-dimensional coordinate gives:

$$\frac{d^4w}{dR^4}=\frac{ql^4}{EIc_3} \tag{33}$$

where: $c_3=(1+c_2+8c_18c_1+4c_1c_2)$. Similarly, Substituting equations (30) and (31) into equation (22) gives:

$$\frac{d^4w}{dR^4}-\frac{m\lambda^2wl^4}{EIc_3}=0 \tag{34}$$

— Determination of Displacement Functions

Solving equations (33) and (34) respectively by open integration yields:

$$w=\sum_{n=0}^{n=4}\frac{a_n}{n!}R^n=A\,h \tag{35}$$

$$w=\sum_{n=0}^{n=\infty}\frac{a_n}{n!}R^n=a\,H \tag{36}$$

For tractable solution and fulfilling the fourth order governing equation of equation (34), the infinite series of equation (36) is truncated and gives:

$$w=\sum_{n=0}^{n=4}\frac{a_n}{n!}R^n=A\,h \tag{37}$$

Substituting equation (37) into equation (31) gives:

$$\theta_s=\frac{B}{L}\frac{dh}{dR} \tag{38}$$

— Formulas for Analyses

The minimization of the total potential energy functional gives the direct governing equation from which the formulas are obtained. Substituting equations (37) into equation (17) and writing the outcome in a more symbolized form gives:

$$\Pi=\frac{EI}{2L^3}\left([A^2+2AB+B^2]k_1+\frac{6\alpha^2}{\{1+\mu\}}[4A^2+4AB+B^2]k_2\right)-\frac{Lm\lambda^2}{2}A^2k_\lambda-AqLk_q \tag{39}$$

where:

$$k_1=\int_0^1\left[\frac{d^2h}{dR^2}\right]^2dR,\ k_2=\int_0^1\left[\frac{dh}{dR}\right]^2dR,\qquad k_\lambda=\int_0^1 h^2\,dR,\qquad k_q=\int_0^1 h\,dR$$

Minimization equation (39) in turn with respect to A and B and yields two simultaneous equations as follow:

$$\frac{d\Pi}{dA}=\left([A+B]k_1+\frac{6\alpha^2}{\{1+\mu\}}[4A+2B]k_2\right)-\frac{AL^4m\lambda^2}{EI}k_\lambda-\frac{qL^4}{EI}k_q=0 \tag{40}$$

$$\frac{d\Pi}{dB} = \frac{E\,I}{2L^3}\left([2A + 2B]k_1 + \frac{6\alpha^2}{\{1+\mu\}}[4A + 2B]k_2\right) - 0 = 0. \tag{41}$$

Solving equations (40) and (41) for pure bending yields:

$$A = \left[\frac{1}{k_1} + \frac{1}{k_2}\left(\frac{1+\mu}{6\alpha^2}\right)\right] \times k_q\beta\left(\frac{qL^4}{EI}\right) \tag{42}$$

$$B = -\left[\frac{2}{k_1} + \frac{1}{k_2}\frac{\{1+\mu\}}{6\alpha^2}\right] \times k_q\beta\left(\frac{qL^4}{EI}\right) \tag{43}$$

Also, solving equations (40) and (41) for free vibration yields:

$$\omega^2 = \frac{6\alpha^2 k_1 k_2}{[k_1\{1+\mu\} + 6\alpha^2 k_2]k_\lambda}\left(\frac{EI}{m\,L^4}\right) \tag{44}$$

— Numerical Example

This example is carried out to demonstrate the applicability of the present theory. The centre deflection w_c and the fundamental natural frequency ω of a simply supported thick isotropic beam of rectangular cross section with Poisson's ratio of 0.25 were determined using the present approach and the results were compared with those of other theories.

After satisfying the boundary conditions, the displacement equations for a simply supported thick isotropic beam are:

$$w = A(R - 2R^3 + R^4) \tag{45}$$

$$\theta_s = \frac{B}{L}(1 - 6R^2 + 4R^3) \tag{46}$$

The centre displacement and the natural fundamental frequency are obtained based on the above solution as:

$$w_c = \beta\left(\frac{qL^4}{EI}\right)\left[\frac{5}{384} + \frac{91}{3264\alpha^2}\right] \tag{47}$$

$$\omega^2 = \frac{284\frac{44}{155}\alpha^2}{\left[\frac{156}{25} + \frac{102}{35}\alpha^2\right]}\left(\frac{EI}{m\,L^4}\right) \tag{48}$$

3. RESULTS AND DISCUSSION

The centre displacement and the natural fundamental frequency are presented in the following non-dimensional form:

$$\overline{w_c} = w\left(\frac{Ebh^3}{qL^4}\right);\ \bar{\omega} = \omega\left(\frac{L^2}{t}\right)\sqrt{\frac{\rho}{E}} \tag{49}$$

The results of the centre displacement and the natural fundamental frequency obtained from the present method vis-avis other theories are presented in tables 1 and 2 respectively.

The results obtained from the present theories are compared with the elementary theory of beam (ETB), first order shear deformation theory (FSDT) of Timoshenko, higher order shear deformation theories of Heyliger and Reddy, Ghugal and exact elasticity solutions given by Timoshenko and Goodier and Cowper. The results of the aforementioned theories are obtained from the work of Sayyad A. S (2011). This is shown in Table 1 and Table 2. The result of the present study gives results that are in close agreement with previous theories except that of the Bernouli-Euler theory. The results of the present study under-predicts the centre deflection when compared with the exact solution, with the highest percentage difference noticed for the beam with a span-to-depth ratio of 2. However, as the span-to-depth ratio increases, the results get closer to the exact solution. The Elementary Beam Theory (ETB) is observed to underestimate the centre displacement across all span-to-depth ratios. The wide difference of the results of the ETB and the results of other studies is because the ETB does not consider shear deformation effects.

This excellent agreement of the results of the present study shows that it is an easy and dependable method that is adequate for the analysis of the free and forced vibration analysis of deep prismatic beams.

4. CONCLUSION

The results of the centre displacement and the natural fundamental frequency of vibration of a deep prismatic beam using modified refined beam theory have been presented. This present study however, modified the works of RBT2 by providing a simple method of obtaining the displacement function which applies to any boundary condition.

The results of an illustrative example showed excellent agreement with results of other theories. However, there was a higher percentage difference in the results obtained when compared with the thin beam theory. The results, however, showed that the percentage difference reduced as the span-depth ratio increased. This small

value of percentage difference show that this present method is a reliable method for the analysis of deep prismatic beams of any given boundary condition and aspect ratio.

Table 1: Centre displacement w_c (at L/2, z = 0) for a simply supported beam

α	Theory	$w\left(\frac{Ebh^3}{qL^4}\right)$	% Difference
2	Present Study	2.399	0.000
	Model 1 (Ambartsumian)	2.357	1.747
	Model 2 (Kruszewski)	2.515	-4.840
	Model 3 (Reddy)	2.523	-5.173
	Model 4 (Touratier)	2.529	-5.423
	Model 5 (Soldatos)	2.513	-4.756
	Model 6 (Karama et. al)	2.51	-4.631
	Model 7 (Akavei)	2.523	-5.799
	Timoshenko(FSDT)	2.538	-5.799
	Bernouli-Euler(ETB)	1.563	34.845
	Timoshenko and Goodier (Exact)	2.453	-2.255
4	Present Study	1.772	0.0000
	Model 1 (Ambartsumian)	1.762	0.5418
	Model 2 (Kruszewski)	1.805	-1.8853
	Model 3 (Reddy)	1.806	-1.9418
	Model 4 (Touratier)	1.805	-1.8853
	Model 5 (Soldatos)	1.802	-1.7160
	Model 6 (Karama et. al)	1.801	-1.6596
	Model 7 (Akavei)	1.804	-1.8289
	Timoshenko(FSDT)	1.806	-1.9418
	Bernouli-Euler(ETB)	1.563	11.7746
	Timoshenko and Goodier (Exact)	1.785	-0.7564
10	Present Study	1.596	0.0000
	Model 1 (Ambartsumian)	1.595	0.0599
	Model 2 (Kruszewski)	1.602	-0.3787
	Model 3 (Reddy)	1.602	-0.3787
	Model 4 (Touratier)	1.601	-0.3161
	Model 5 (Soldatos)	1.601	-0.3161
	Model 6 (Karama et. al)	1.601	-0.3161
	Model 7 (Akavei)	1.601	-0.3161
	Timoshenko(FSDT)	1.602	-0.3787
	Bernouli-Euler(ETB)	1.563	2.0650
	Timoshenko and Goodier(Exact)	1.598	-0.1281

Table 2: Resonating frequency w, for a simply supported beam

α	Theory	$\omega\left(\frac{L^2}{t}\right)\sqrt{\frac{\rho}{E}}$	% difference
4	Present Study	2.677	0.000
	Model 1 (Ambartsumian)	2.625	1.96467
	Model 2 (Kruszewski)	2.597	3.01038
	Model 3 (Reddy)	2.596	3.047727
	Model 4 (Touratier)	2.596	3.047727
	Model 5 (Soldatos)	2.596	3.047727
	Model 6 (Karama et. al)	2.608	2.599565
	Model 7 (Akavei)	2.598	2.973033
	Bernouli-Euler(ETB)	2.779	-3.78674
	Timoshenko (FSDT)	2.624	2.002016
	Ghugal	2.602	2.823646
	Heyliger and Reddy	2.596	3.047727
	Cowper	2.602	2.823646
10	Present Study	2.821	0.000
	Model 1 (Ambartsumian)	2.808	0.464476
	Model 2 (Kruszewski)	2.802	0.677159
	Model 3 (Reddy)	2.802	0.677159
	Model 4 (Touratier)	2.802	0.677159
	Model 5 (Soldatos)	2.802	0.677159
	Model 6 (Karama et. al)	2.805	0.570817
	Model 7 (Akavei)	2.803	0.641711
	Bernouli-Euler(ETB)	2.838	-0.59894
	Timoshenko (FSDT)	2.808	0.464476
	Ghugal	2.804	0.606264
	Heyliger and Reddy	2.802	0.677159
	Cowper	2.804	0.606264

The close agreement of the results of the present formulation with those obtained from other theories showed that the present formulation can be used as a useful tool for the analysis of the free and forced vibration analysis of deep prismatic beams.

References

[1] Cowper, G. R. On the accuracy of Timoshenko's beam theory, ASCE Journal of Engineering Mechanics Division 94 (6), 1968, pp 447–1 453.
[2] Gawali, A.L. and Sanjay, C.K. Vibration analysis of beams. World research Journal Of Civil Engineering, Vol. 1, Issue 1, 2011, pp 15- 29.
[3] Ghugal, Y. M. A simple higher order theory for beam with transverse shear and transverse normal effect, Department Report 4, Applied of Mechanics Department, Government College of Engineering, Aurangabad, India, 2006.
[4] Heyliger, P. R., Reddy, J. N. A higher order beam finite element for bending and vibration problems. Journal of Sound and Vibration 126 (2), 1988, pp 309–326.
[5] Ibearugbulem, O.M., Njoku, K.O., Ibearugbulem, C.N., Onuoha, I. Correct assumption approach to refined theory for analysis of thick beam. International Journal of Advanced research (IJOAR), Vol.3, Issue 10, 2015, pp 1-10.
[6] Lamb, H. On waves in an elastic plates. Proceeding of Royal Society London, Series A. 93, 1917, Pp 114–128.
[7] Mesut, S. and Turgut, K. Free vibration analysis of beams using a third-order shear deformation theory, 2007.
[8] Sayyad, A. S. Comparison of various refined beam theories for the bending and free vibration analysis of thick beams. Journal of Applied and Computational Mechanics, Vol. 5, 2011, pp 217–230.
[9] Timoshenko, S. P. On the correction for shear of the differential equation for transverse vibrations of prismatic bars. Philosophical Magazine 41 (6), 1921, pp 742–746.
[10] Timoshenko, S. P., Goodier, J. N. Theory of Elasticity, McGraw-Hill 3rd Int. Ed. Singapore, 1970.
[11] Vikas, A. and Ajay, G.D. Bending analysis of deep beam using refined shear deformation theory. International journal of engineering research, Vol. 5, Issue 3, 2016, pp 526-531.
[12] Wang, C.M., Reddy, C.N. and Lee, K.H. Shear deformable beams and plate, Relationship with classical solution. Amsterdam: Elsevier Science Ltd, 2000.