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ELECTRICAL RESISTANCE OF NON-HOMOGENEOUS CONICAL BODY

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Abstract: The aim of this paper is to determine the electric resistance of a hollow conical conductor body for the steady-state current flow. The studied steady-state conduction problem is axisymmetric. The material of the conductor body is isotropic and non-homogeneous. The conductivity is a smooth function of the polar angle of a spherical coordinate system. The materials, whose properties are smooth functions of the space coordinates, are called functionally graded materials. An analytical solution which is based on the governing equations of steady-state problems of electricity is presented. The effect of the non-homogeneity to the electric resistance is analyzed. Two types of non-homogeneities are considered. The numerical results of the paper can be used as benchmark solutions for the usual numerical methods, such as finite element method finite difference method, boundary element method, etc.

Keywords: electrical resistance, steady-state current flow, non-homogeneity

1. INTRODUCTION

Electrical resistance of an electrical conductor is the measure of the difficulty to pass a steady electric current through that conductor body. The definition of the electrical resistance is based on Ohm's law it is defined as the ratio of the applied voltage to the current. In paper [8], upper and lower bounds are proven for the electrical resistance of homogeneous ring like axisymmetric conductor. Paper by Ecsedi and Baksa are developed a mathematical model to determine the steady-state electric current flow through in non-homogeneous isotropic conductor whose shape has a three-dimensional hollow body [9]. This paper deals with the determination of the electric resistance of a conical body which bordered by two conical surfaces and two spherical surfaces as shown in Figure 1. The apex of conical surfaces and the center of spherical surfaces are the same points (Figures 1, 2). The spherical coordinate system $Or\varphi\vartheta$ will be used to formulate the governing equations. The connection of the rectangular coordinates x, y, z and spherical coordinates r, φ, ϑ is as follows (Figure 1)

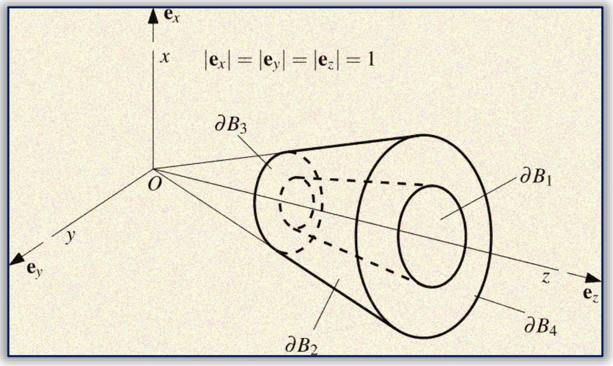

Figure 1. Non-homogeneous conical conductor body

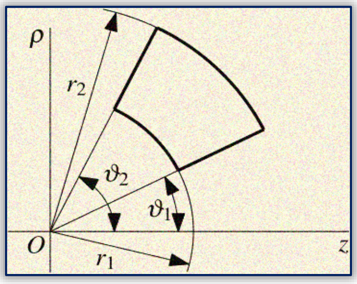

Figure 2. Meridian section of the conical body

$$x = r\cos\varphi\sin\vartheta, \quad y = r\sin\varphi\sin\vartheta, \quad z = r\cos\vartheta. \tag{1}$$

The space domain occupied by conical body B in the spherical coordinates can be given as

$$B = \{(r,\varphi,\vartheta) \,|\, r_1 \le r \le r_2,\ 0 \le \varphi \le 2\pi,\ 0 \le \vartheta \le \pi\}. \tag{2}$$

Later on we will use next formulae [1]

$$\nabla = \frac{\partial}{\partial r}\mathbf{e}_r + \frac{1}{r\sin\vartheta}\frac{\partial}{\partial\varphi}\mathbf{e}_\varphi + \frac{1}{r}\frac{\partial}{\partial\vartheta}\mathbf{e}_\vartheta, \tag{3}$$

$$\nabla F = \frac{\partial F}{\partial r}\mathbf{e}_r + \frac{1}{r\sin\vartheta}\frac{\partial F}{\partial\varphi}\mathbf{e}_\varphi + \frac{1}{r}\frac{\partial F}{\partial\vartheta}\mathbf{e}_\vartheta, \tag{4}$$

$$\Delta F = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial F}{\partial r}\right) + \frac{1}{r^2\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial F}{\partial\vartheta}\right) + \frac{1}{r^2\sin^2\vartheta}\frac{\partial^2 F}{\partial\varphi^2}. \tag{5}$$

The unit vectors of the spherical coordinate system $Or\varphi\vartheta$ are denoted by $\mathbf{e}_r, \mathbf{e}_\varphi, \mathbf{e}_\vartheta$ [1]. The boundary surfaces of conductor body B are separated into four different parts as $\partial B = \partial B_1 \cup \partial B_2 \cup \partial B_3 \cup \partial B_4$ where

$$\partial B_1 = \{(r,\varphi,\vartheta) \,|\, r_1 \le r \le r_2,\ 0 \le \varphi \le 2\pi,\ \vartheta = \vartheta_1\}, \tag{6}$$

$$\partial B_2 = \{(r,\varphi,\vartheta) \,|\, r_1 \le r \le r_2,\ 0 \le \varphi \le 2\pi,\ \vartheta = \vartheta_2\}, \tag{7}$$

$$\partial B_3 = \{(r,\varphi,\vartheta) \,|\, r = r_1,\ 0 \le \varphi \le 2\pi,\ \vartheta_1 \le \vartheta \le \vartheta_2\}, \tag{8}$$

$$\partial B_4 = \{(r,\varphi,\vartheta) \,|\, r = r_2,\ 0 \le \varphi \le 2\pi,\ \vartheta_1 \le \vartheta \le \vartheta\}. \tag{9}$$

The boundary surface segments ∂B_3 and ∂B_4 are insulated and on the boundary surface segments ∂B_1 and ∂B_2 the electric potential $U = U(r,\varphi,\vartheta)$ is prescribed that is

$$U(r,\varphi,\vartheta) = U_1 = \text{constant on } \partial B_1, \tag{10}$$

$$U(r,\varphi,\vartheta) = U_2 = \text{constant on } \partial B_2, \tag{11}$$

where $U_1 \neq U_2$. The boundary conditions on the insulated boundary surface segment in terms of $U = U(r,\varphi,\vartheta)$ can be formulated as

$$\mathbf{n}\cdot\nabla U = \frac{\partial U}{\partial n} = \frac{\partial U}{\partial r} = 0 \text{ on } \partial B_3 \cup \partial B_4. \tag{12}$$

Here, $\mathbf{n}$ is the unit normal vector to the insulated surfaces ∂B_3 and ∂B_4 that is $\mathbf{n} = \mathbf{e}_r$. The dot between two vectors denotes the scalar product. According to theory of the steady-state current flow we have the next equations for the steady motion of charges [2,3,4,5,7]

$$\mathbf{j} = \sigma\mathbf{E}, \quad \nabla\cdot\mathbf{j} = 0, \quad \mathbf{E} = -\nabla U. \tag{13}$$

In Eq. $(13)_1$ $\sigma = \sigma(r,\varphi,\vartheta)$ denotes the conductivity of the isotropic non-homogeneous conductor body [3,4]. The International System of Units (SI) is used throughout in this paper. Ohm's law $(13)_1$ is based on the experimental observation which formulates that at constant temperature in isotropic conductor the current density vector $\mathbf{j}$ is proportional to the electric field vector $\mathbf{E}$. From the above equations it follows that

$$\nabla\cdot(\sigma\nabla U) = 0, \quad (r,\varphi,\vartheta)\in B, \tag{14}$$

$$\mathbf{j}\cdot\mathbf{n} = -\sigma\mathbf{n}\cdot\nabla U = -\sigma\frac{\partial U}{\partial n} = -\sigma\frac{\partial U}{\partial r} = 0, \quad (r,\varphi,\vartheta)\in \partial B_3\cup\partial B_4, \tag{15}$$

that is on the insulated boundary surface segments the boundary condition (12) is valid. We introduce a new function $u = u(r,\varphi,\vartheta)$ by the next definition

$$U(r,\varphi,\vartheta) = (U_2 - U_1)u(r,\varphi,\vartheta) + U_1. \tag{16}$$

It is evident $u = u(r,\varphi,\vartheta)$ satisfies the next boundary-value problem

$$\sigma\Delta u + \nabla\sigma\cdot\nabla u = 0, \quad (r,\varphi,\vartheta)\in B, \tag{17}$$

$$u(r,\varphi,\vartheta) = 0, \quad (r,\varphi,\vartheta)\in\partial B_1, \tag{18}$$

$$u(r,\varphi,\vartheta) = 0, \quad (r,\varphi,\vartheta)\in\partial B_2, \tag{19}$$

$$\frac{\partial u}{\partial r} = 0, \quad (r,\varphi,\vartheta)\in\partial B_3\cup\partial B_4. \tag{20}$$

Here we note, $u = u(r,\varphi,\vartheta)$ is unit free. An electric current in the conductor is the continuous passage of electric charges along the conductor. The constant electric potential difference between the surfaces ∂B_1 and ∂B_2 maintain the steady flow of the electric current. The amount of charge following through surface segment ∂B_2 per unit time is denoted by I. The determination of the current I is based on the next equation [2,3]

$$I = \int_{\partial B_2}\mathbf{j}\cdot\mathbf{n}\mathrm{d}A = -\int_{\partial B_2}\sigma\mathbf{n}\cdot\nabla U\mathrm{d}A = (U_1 - U_2)\int_{\partial B_2}\sigma\frac{\partial u}{\partial n}\mathrm{d}A. \tag{21}$$

Here, $\mathrm{d}A$ denotes the area element and in the present problem

$$\frac{\partial u}{\partial n} = \mathbf{n}\cdot\nabla u = \mathbf{e}_\vartheta\cdot\nabla u = \frac{1}{r}\frac{\partial u}{\partial\vartheta}. \tag{22}$$

The considered problem is axisymmetric since we assume that $\sigma = \sigma(\vartheta)$ and we have $u = u(r,\vartheta)$. The expression of area element of surface segment ∂B_2 is

$$\mathrm{d}A = 2\pi r\sin\vartheta_2\mathrm{d}r. \tag{23}$$

Substitution of Eqs. (22) and (23) into the Eq. (21) gives

$$I = (U_1 - U_2)2\pi\sin\vartheta_2\sigma(\vartheta_2)\int_{r_1}^{r_2}\left.\frac{\partial u}{\partial\vartheta}\right|_{\vartheta_2}\mathrm{d}r. \tag{24}$$

The electrical resistance R of the conductor body is defined as

$$R=\frac{U_1-U_2}{I}=\frac{1}{2\pi\sin\vartheta_2\sigma\left(\vartheta_2\right)\int_{r_1}^{r_2}\left.\frac{\partial u}{\partial\vartheta}\right|_{\vartheta_2}\mathrm{d}r}. \tag{25}$$

2. DETERMINATION OF ELECTRICAL RESISTANCE

From Eq. (17) it follows that

$$\sigma\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)+\frac{1}{r^2}\frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial u}{\partial\vartheta}\right)\right]+\frac{1}{r^2}\frac{\partial\sigma}{\partial\vartheta}\frac{\partial u}{\partial\vartheta}=0,\quad(r,\varphi,\vartheta)\in B, \tag{26}$$

Since we have $\sigma=\sigma(\vartheta)$ and the considered problem is axisymmetric. Next we assume $u=u(\vartheta)$. According to boundary conditions (18) and (19) and Eq. (26) we get the function $u=u(\vartheta)$ is the solution of the next boundary-value problem

$$\frac{\sigma(\vartheta)}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial u}{\partial\vartheta}\right)+\frac{\partial\sigma}{\partial\vartheta}\frac{\partial u}{\partial\vartheta}=0,\quad(r,\varphi,\vartheta)\in B, \tag{27}$$

$$u\left(\vartheta_1\right)=0,\quad(r,\varphi,\vartheta)\in\partial B_1,\quad u\left(\vartheta_2\right)=1,\quad(r,\varphi,\vartheta)\in\partial B_2. \tag{28}$$

It is evident that $u=u(\vartheta)$ satisfies the boundary condition (12). Solution of the second order ordinary differential equation for $u=u(\vartheta)$ under the boundary condition (28) is as follows

$$u(\vartheta)=\frac{\int_{\vartheta_1}^{\vartheta}\frac{\mathrm{d}\xi}{\sigma(\xi)\sin\xi}}{\int_{\vartheta_1}^{\vartheta_2}\frac{\mathrm{d}\vartheta}{\sigma(\vartheta)\sin\vartheta}},\quad\vartheta_1\le\vartheta\le\vartheta_2. \tag{29}$$

From Eq. (29) it follows that

$$\sigma(\vartheta)\sin\vartheta\frac{\partial u}{\partial\vartheta}=\frac{1}{\int_{\vartheta_1}^{\vartheta_2}\frac{\mathrm{d}\vartheta}{\sigma(\vartheta)\sin\vartheta}}. \tag{30}$$

Substitution Eq. (30) into the formula of electrical resistance gives

$$R=\frac{\int_{\vartheta_1}^{\vartheta_2}\frac{\mathrm{d}\vartheta}{\sigma(\vartheta)\sin\vartheta}}{2\pi\left(r_2-r_1\right)}. \tag{31}$$

3. EXAMPLES FOR CONICAL CONDUCTOR

For conical conductor body which has the next data $r_1=0.1\text{ m},\ r_2=0.5\text{ m},\ \vartheta_1=\pi/8,\ \vartheta_2=\pi/3$ the electric resistance is determined if

Case a)

$$\sigma(\vartheta)=\sigma_0\left(\frac{\vartheta}{\vartheta_1}\right)^n, \tag{32}$$

Case b)

$$\sigma(\vartheta)=\sigma_0\exp\left(\alpha\vartheta\right). \tag{33}$$

Here,

$$\sigma_0=3.77\times10^7\ \frac{\text{A}}{\text{Vm}}. \tag{34}$$

Figure 3 shows the dependence of electric resistance from power index n (case a). The plot of electric resistance as a function of α is presented in Figure 4.

4. RADIALLY NON-HOMOGENEOUS CIRCULAR CYLINDER

When the point O is an infinite distance point then the conical surfaces will be circular cylinder whose creators are parallel to axis z and the spherical surfaces will be perpendicular planes to the common axis of boundary circular cylinder (Figure 5). Assuming radial non-homogeneity of the circular cylindrical conductor by the same method as was used in Sections 1 and 2 of this paper we can compute the electric resistance of this conductor. The end cross sections at $z=0$ and $z=L$ are insulated and the inner and outer cylindrical boundary surfaces have given electric potential U_1 and U_2 (Figure 5).

The space domain B occupied by the hollow circular cylinder in the polar coordinate system $Or\varphi z$ can be given as

$$B=\{(r,\varphi,z)\,|\,r_1\le r\le r_2,\ 0\le\varphi\le 2\pi,\ 0\le z\le L\}. \tag{35}$$

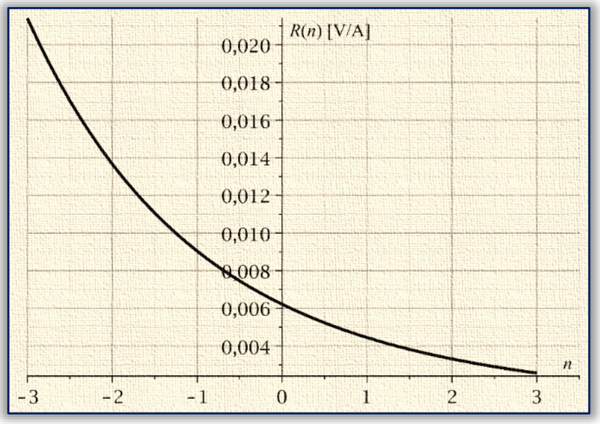

Figure 3. Resistance of the conical conductor body as a function of power index n, case a)

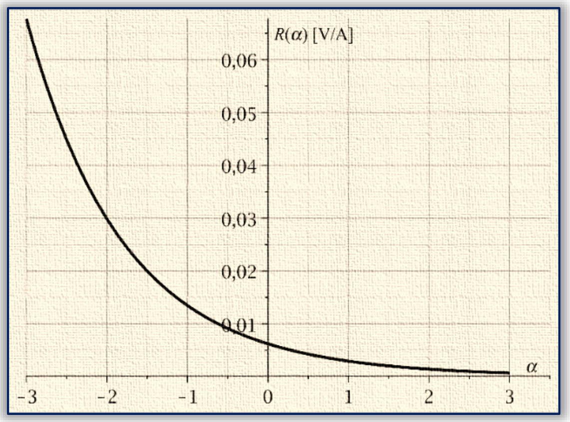

Figure 4. Resistance of the conical conductor body as a function α, case b)

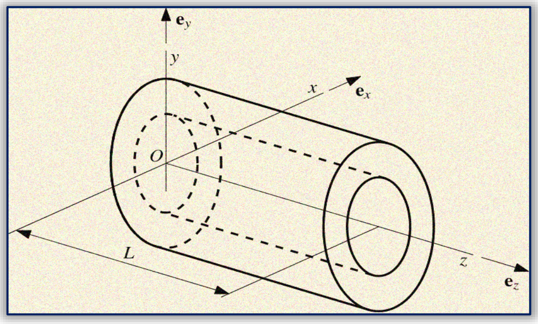

Figure 5. Radially non-homogeneous circular cylinder

The boundary surface of B is

$$\partial B_1=\{(r,\varphi,z)\,|\,r=r_1,\ 0\le\varphi\le 2\pi,\ 0\le z\le L\}, \tag{36}$$

$$\partial B_2=\{(r,\varphi,z)\,|\,r=r_2,\ 0\le\varphi\le 2\pi,\ 0\le z\le L\}, \tag{37}$$

$$\partial B_3=\{(r,\varphi,z)\,|\,r_1\le r\le r_2,\ 0\le\varphi\le 2\pi,\ z=0\}, \tag{38}$$

$$\partial B_4=\{(r,\varphi,z)\,|\,r_1\le r\le r_2,\ 0\le\varphi\le 2\pi,\ z=L\}, \tag{39}$$

We assume that the electric potential U depends only on the radial coordinate r that is $U=U(r)$. A new function $u=u(r)$ will be introduced by the next definition:

$$U(r)=(U_2-U_1)u(r)+U_1. \tag{40}$$

By the use of Eqs. $(14)_{1,2,3}$ it can be shown that $u=u(r)$ is the solution of the next boundary-value problem

$$\sigma(r)\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}\right)+\frac{\partial\sigma}{\partial r}\frac{\partial u}{\partial r}=0,\quad r_1\le r\le r_2, \tag{41}$$

$$u(r_1)=0,\quad u(r_2)=1. \tag{42}$$

The boundary conditions on the insulated boundary surface segments ∂B_3 and ∂B_4 are satisfied, since $u=u(r)$ does not depend on the axial coordinate z. From Eqs. (41), (42) it follows that

$$u(r)=\frac{\displaystyle\int_{r_1}^{r}\frac{\mathrm{d}\rho}{\rho\sigma(\rho)}}{\displaystyle\int_{r_1}^{r_2}\frac{\mathrm{d}\rho}{\rho\sigma(\rho)}},\quad r_1\le r\le r_2. \tag{43}$$

Simple computation shows that

$$r\sigma(r)\frac{\partial u}{\partial r}=\frac{1}{\displaystyle\int_{r_1}^{r_2}\frac{\mathrm{d}r}{r\sigma(r)}}. \tag{44}$$

It can be proven that the connection between the potential difference U_2-U_1 and the current I is formulated as

$$I=\frac{2\pi L(U_2-U_1)}{\displaystyle\int_{r_1}^{r_2}\frac{\mathrm{d}r}{r\sigma(r)}}. \tag{45}$$

From Eq. (45) we can get immediately the expressions of electric resistance R of the radially non-homogeneous hollow cylindrical conductor

$$R = \frac{\int_{r_1}^{r_2} \frac{\mathrm{d}r}{r\sigma(r)}}{2\pi L}. \tag{46}$$

5. EXAMPLE FOR HOLLOW CIRCULAR CYLINDRICAL CONDUCTOR

For hollow non-homogeneous circular cylinder two types of radial non-homogeneity will be considered
Case (a) power law radial non-homogeneity

$$\sigma_r = \sigma_0 \left(\frac{r}{r_1}\right)^n, \quad \sigma_0 = \text{constant}, \tag{47}$$

Case (b) exponential radial non-homogeneity

$$\sigma(r) = \sigma_0 \exp(\alpha r), \quad \sigma_0 = \text{constant}. \tag{48}$$

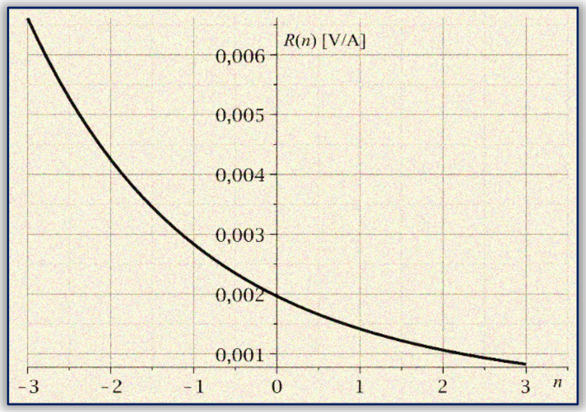

Figure 6. The graph of function $R = R(n)$ for power radial non-homogeneity

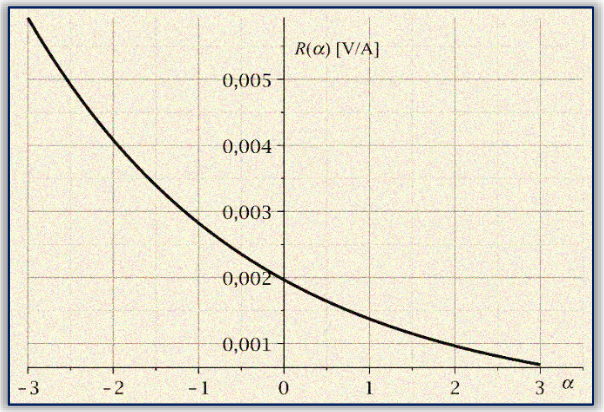

Figure 7. The graph of function $R = R(\alpha)$ for exponential radial non-homogeneity

In Eqs. (47), (48) n and α are material parameters they describe the degree of the material inhomogeneity. For case (a) we obtain from the formula (46)

$$R(n) = \frac{1 - \left(\frac{r_1}{r_2}\right)^n}{2\pi\sigma_0 n L}. \tag{49}$$

For case (b) we have

$$R(\alpha) = \frac{\int_{r_1}^{r_2} \frac{1}{r\exp(\alpha r)} \mathrm{d}r}{2\pi\sigma_0 L} = \frac{Ei(1, \alpha r_1) - Ei(1, \alpha r_2)}{2\pi\sigma_0 L}. \tag{50}$$

In Eq. (50) [6]

$$Ei(1, x) = -\int \frac{\mathrm{d}x}{x\exp(x)}. \tag{51}$$

In the case of power radial inhomogeneity Figure 6 shows the graph of function $R = R(n)$. The dependence of electrical resistance from the material parameter α in the case (b) is presented in Figure 7. The following numerical data are used in plots of $R = R(n)$ and $R(\alpha)$: $r_1 = 0.25$ m, $r_2 = 0.5$ m, $L = 1$ m, $\sigma_0 = \frac{1}{\rho_0}$, $\rho_0 = 0.0178 \ \frac{\mathrm{Vm}}{\mathrm{A}}$.

6. CONCLUSIONS

In this paper the electric resistance of a conical isotropic inhomogeneous conductor is determined by the use of governing equations of steady state current flow of electricity. Analytical closed form solutions are presented for the electric potentials as a function position and the expressions of electric resistances. The effects of non-homogeneity for power law and exponential law types of non-homogeneities are analyzed. Two numerical examples illustrate the applications of the formulated exact solutions. The case of the radially non-homogeneous hollow circular cylinder is also examined.

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