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SOLVING NONLINEAR EQUATIONS USING NEW ITERATIVE SCHEME

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Abstract: This paper proposes new single–step iterative scheme for finding root of a nonlinear equation $f(x) = 0$ by considering fixed point x_p on the x–axis as well as initial guess value x_0. The scheme is derived by using the concepts of similarity of triangles and Taylor's series expansion. The convergence analysis shows that the proposed method has linear convergence, but the convergence rate can be further improved nearer to order two for limiting case. The advantage of the proposed scheme is that, it works even if $f'(x) = 0$, which is the limitation of the Newton's method as well as the methods suggested by different authors. Some examples are discussed and compared with Newton's method. From the results, it is found that the proposed method behaves same as Newton's method for same initial guess value. The number of iterations and solutions obtained are also remains the same. A special case (example) is also discussed where it is shown that the proposed method works even if Newton's method fails.
Keywords: Iterative method, Convergence, Nonlinear equations, Newton's method

1. INTRODUCTION

Solving nonlinear equations of the form $f(x) = 0$ always have an attraction for researchers due to its applications in different disciplines of science and engineering. Newton's method is the most popular iterative method for solving nonlinear equations having second–order convergence. Many researchers [1–13] developed new iterative schemes for better and faster convergence. Shah and Shah [12, 13] also suggested new iterative methods for solving nonlinear equations.

This paper proposes new single–step iterative scheme for finding root of a nonlinear equation by considering fixed point x_p on the x–axis as well as initial guess value x_0, using the concepts of similarity of triangles and Taylor's series expansion. The convergence analysis shows that the proposed method has linear convergence, but the convergence rate can be further improved nearer to order two for limiting case. The advantage of the proposed scheme is that, it works even if $f'(x) = 0$. Some examples are discussed and compared with Newton's method. The limitations of the proposed method are also discussed.

2. THE PROPOSED METHOD

Figure 1 shows the geometric view of the proposed method for finding root of a nonlinear equation $f(x) = 0$. Assume that $f(x), f'(x)$ and $f''(x)$ are continuous nearer to exact root x^*, where x^* is a simple root in some open interval $I \subset R$. Let $A(x_p, 0)$ be the fixed point on the x–axis and $B(x_0, 0)$ be the initial guess value sufficiently close to x^*. Let $C(x_0, f(x_0))$ be the point lie on the curve $y = f(x)$ as shown in figure. Let L_1 be a line joining the points B and C, then $D(x_0, kf(x_0))$ be a point on L_1 and inside the line segment $\overline{BC}$ for $0 < k < 1$. Draw a line l_1 joining the points A and D, which intersect the curve $y = f(x)$ at $E(x_1, f(x_1))$ (say), where $x_1 = x_0 + h$, giving the first approximation x_1 (call it $F(x_1, 0)$) by the following procedure.

As shown in figure, ΔAFE and ΔABD are similar triangles, therefore

$$\frac{FE}{AF} = \frac{BD}{AB}$$

implies

$$\frac{f(x_1)}{x_1 - x_p} = \frac{kf(x_0)}{x_0 - x_p}. \tag{1}$$

Using the fact $x_1 = x_0 + h$ and Taylor series $f(x_1) = f(x_0 + h) = f(x_0) + hf'(x_0)$, (1) becomes

$$x_1 = \frac{(x_0 - x_p)[f(x_0) - x_0 f'(x_0)] + kx_p f(x_0)}{kf(x_0) - (x_0 - x_p)f'(x_0)}.$$

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By repeating the above procedure, a sequence of approximations $\{x_{n+1}\}$ for n≥1 is obtained from the following general formula

$$x_{n+1} = \frac{(x_n - x_p)[f(x_n) - x_n f'(x_n)] + kx_p f(x_n)}{kf(x_n) - (x_n - x_p)f'(x_n)}; 0 < k < 1, \qquad (2)$$

which converges to the exact root x^*, provided $kf(x_n) - (x_n - x_p)f'(x_n) \neq 0$.

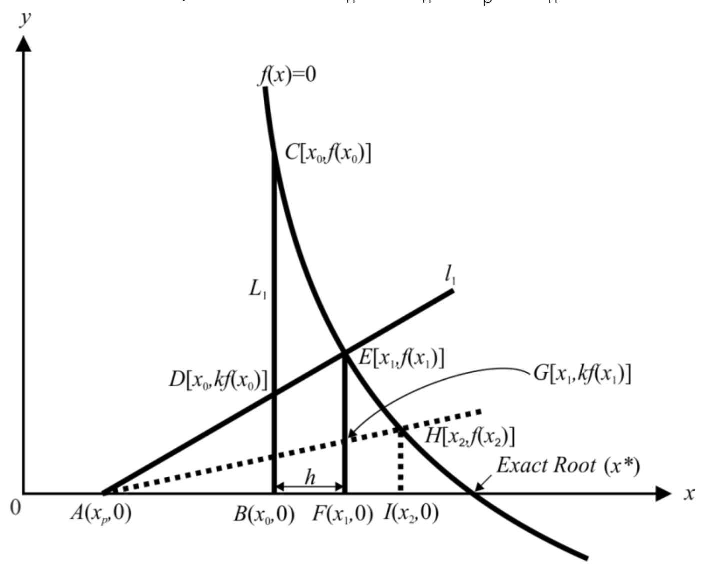

Figure 1. The proposed method

3. ORDER OF CONVERGENCE

Let ε_n be the error in the n^{th} – iteration and ε_{n+1} be the error in the $(n+1)^{th}$ – iteration, then

$$\varepsilon_n = x^* - x_n, \varepsilon_{n+1} = x^* - x_{n+1}.$$

On substituting, equation (2) implies

$$\varepsilon_{n+1} \approx k\varepsilon_n + (\text{Finite Quantity})\ \varepsilon_n^2$$

using the fact that $f(x^*) = 0$.

Thus, the method converges linearly. However, it is observed that when $k \to 0$, the order of convergence can be improved to order two and the formula (2) becomes Newton's formula as shown below.

$$\lim_{k\to 0} x_{n+1} = \lim_{k\to 0} \frac{(x_n - x_p)[f(x_n) - x_n f'(x_n)] + kx_p f(x_n)}{kf(x_n) - (x_n - x_p)f'(x_n)} \Rightarrow x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

4. RESULTS AND DISCUSSION

As the proposed iterative scheme converges linearly and the order of convergence improved to order two when $k \to 0$, so a comparison of the proposed method is given with the Newton's method.

Table 1. Comparative study of the proposed method (2) with the Newton's method for $x^3 - x^2 - x - 1 = 0$

(1) Initial guess value x_0	(2) Value of x_p	(3) Value of k	(4) Number of iterations required by the proposed method (2)	(5) Number of iterations required by the Newton's method	(6) Solution of the given equation by the proposed method (2)	(7) Solution of the given equation by the Newton's method
1.5	0.5	0.00001	4	4	1.839286755214739	1.839286755214164
	– 1.5					
	– 3.0					
	3.0					
	6.0					

In Table 1 initial guess value x_0 is taken after applying intermediate value theorem, but it is not necessary. Accordingly, x_0 = 1.5 is taken because $f(1) < 0$ and $f(2) > 0$. It is seen from the table that, the proposed method gives same result correct up to twelve decimal places as Newton's method when k = 0.00001 and for different values of x_p. The number iterations also remain the same. It is seen from Tables 1 and 2 that, the value of k = 0.00001 works almost good for getting second–order convergence. Table 3 shows that

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the proposed method works even if $f'(x)=0$, which is the limitation of the Newton's method as well as methods suggested by authors [1, 3, 6–10].

Here, it should be noted that the above results are calculated for $|x_{n+1}-x_n|<0.00001$.

Table 2. Comparative study of the proposed method (2) with the Newton's method for different equations

(1) Equations	(2) Initial guess value x_0	(3) Value of x_p	(4) Value of k	(5) Number of iterations required by the proposed method (2)	(6) Number of iterations required by the Newton's method	(7) Solution of the given equation by the proposed method (2)	(8) Solution of the given equation by the Newton's method
$\sin x = 0$	1.5	0.5	0.00001	3	3	– 12.566370614359171	–12.566370614359172
$x^3 + x^2 - 2 = 0$	2.2	0		5	5	1.000000000000073	1.000000000000000
$(x-2)^{23} - 1 = 0$	3.5	– 4		12	12	3.000000000037657	3.000000000022981
$xe^{x^2} - \sin^2 x + 3\cos x + 5 = 0$	– 1	0		4	4	– 1.207647827130997	– 1.207647827130919
$xe^{x^2} - \sin^2 x + 3\cos x + 5 = 0$	– 2	0		6	6	– 1.207647827227377	–1.207647827173531
$xe^{x^2} - \sin^2 x + 3\cos x + 5 = 0$	0	3		69	69	– 1.207647827261265	–1.207647827199518

Table 3. Comparative study of the proposed method (2) with the Newton's method for $x^3 + x^2 - 2 = 0$

(1) Initial guess value x_0	(2) Value of x_p	(3) Value of k	(4) Number of iterations required by the proposed method (2)	(5) Number of iterations required by the Newton's method	(6) Solution of the given equation by the proposed method (2)	(7) Solution of the given equation by the Newton's method
0	2	0.00001	35	Fails	1.000000000078346	Fails

5. CONCLUSIONS

The present paper proposes new single–step iteration scheme for finding root of a nonlinear equation f(x) = 0 using the concepts of similarity of triangles and Taylor's series expansion. From the method and discussion of results, the following conclusions can be drawn.

(1) The proposed method converges linearly. But for smaller values of k, the convergence rate can be improved to order two and the formulae become Newton's formula.

(2) The major advantage of the method is that it works even if $f'(x_n)=0$. Thus, equations which are unable to solve due to this limitation in different methods suggested in [1, 3, 6–10] can be solved by the proposed method, and the same second–order convergence rate can be achieved for smaller values of k.

(3) The condition of validity of the proposed scheme is

$$kf(x_n) - (x_n - x_p)f'(x_n) \neq 0.$$

If $kf(x_n) - (x_n - x_p)f'(x_n) = 0$ is chosen, then

$$kf(x_n) = (x_n - x_p)f'(x_n).$$

But $k \neq 0$ since $0<k<1$, and $(x_n - x_p) \neq 0$ since x_n and x_p are two different points. Hence, $kf(x_n) - (x_n - x_p)f'(x_n) = 0$ is satisfied only if $f(x_n)=0$ and $f'(x_n)=0$ simultaneously.

Again, if $kf(x_n) - (x_n - x_p)f'(x_n) = 0$ is chosen, then

$$\frac{k}{x_n - x_p} = \frac{f'(x_n)}{f(x_n)}.$$

If k and x_p are chosen in such a way that the above condition is satisfied at any stage of iterations, then also the proposed method fails. But this condition achieved rarely.

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