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DETERMINATION OF THE DEFORMATION OF BEAMS UNDER THE ACTION OF AXIAL AND THERMAL LOADS

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Abstract: In recent years, a lot of studies were performed on the mechanics of beams. One of the most important aspects of the computations regarding these problems are the stresses and deformations caused by combined mechanical and thermal loads. This paper deals with the determination of the deformation of homogeneous isotropic linearly elastic beams under the action of axial load and thermal loading. In our problem prismatic a beam is considered, which is subjected to combined thermal and mechanical loads, and our aim is to present the analytical solution to this problem when the temperature field is a specific linear function of the cross sectional coordinates. The applied mechanical load consists of an axial load and bending moment applied them at the end cross sections of the considered prismatic beam. The thermal loading is obtained from a temperature field which is a linear function of the cross–sectional coordinates.
Keywords: beam, axial load, thermoelasticity

1. INTRODUCTION

Beams are well known structural elements that have been used successfully in various engineering applications for their load carrying capabilities. In recent years, a lot of studies were performed on the mechanics of beams. One of the most important aspects of the computations regarding these problems are the stresses and deformations caused by combined mechanical and thermal loads.

A lot of books can be found that describe the fundamental equations of beams, furthermore we have several sources to tackle thermoelastic problems, such as [1–6]. There are a lot of papers dealing with the stability problems of beams (e.g. [7, 8]). Paper [9] presented the basic equations of heterogeneous beams under pure mechanical loading, while [10] used the principle of minimum complementary energy method to tackle the thermoelastic problem of prismatic bars. Work [11] considered the thermoelastic problem of curved beams under pure thermal loading coming from a temperature field.

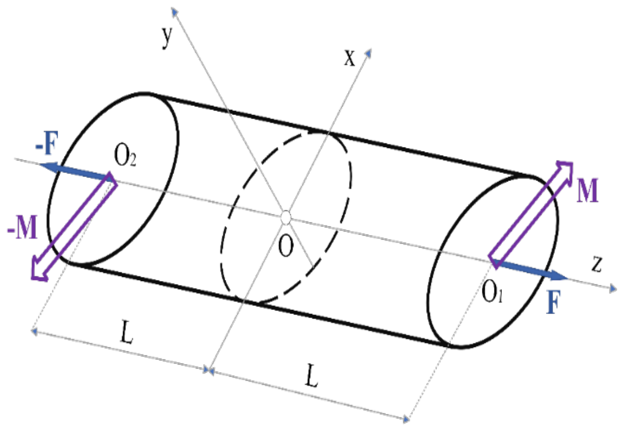

Figure 1. The sketch of the problem

In our problem prismatic a beam is considered, which is subjected to combined thermal and mechanical loads, and our aim is to present the analytical solution to this problem when the temperature field is a specific linear function of the cross sectional coordinates. Figure 1 shows the beams whose cross section is denoted by A, its length is $2L$. The centerline of the homogeneous beam is the axis z. The linear theory of steady–state thermoelasticity is applied. [1–3]. It is assumed that the solution of the considered one–dimensional thermoelasticity problem is based on the following displacement field:

$$\mathbf{u} = u(z)\mathbf{e}_x + v(z)\mathbf{e}_y + w(x,y,z)\mathbf{e}_z, \tag{1}$$

where

$$w(x,y,z) = w_0(z) - x\frac{\partial u}{\partial z} - y\frac{\partial v}{\partial z}. \tag{2}$$

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In Eqs. (1) and (2) *x*, *y* and *z* are the Cartesian coordinates, axis *z* is the center line of the beam. The vectors $\mathbf{e}_x$, $\mathbf{e}_y$ and $\mathbf{e}_z$ are the unit vectors of the Cartesian coordinate system *Oxyz* as shown in Fig. 1. The applied thermal loading can be derived from the following temperature field

$$T(x,y)=\mathring{T}(x,y)-T_0=t_0+xt_x+yt_y, \tag{3}$$

where $\mathring{T}(x,y)$ is the absolute temperature.

The elastic body is in an unstressed state at the absolute temperature T_0 [4, 5] and t_0, t_x, t_y are prescribed constants of the temperature change. Let us consider the next notations

$$\mathbf{R}=x\mathbf{e}_x+y\mathbf{e}_y,$$
$$\mathbf{U}=u(z)\mathbf{e}_x+v(z)\mathbf{e}_y, \tag{4}$$
$$\mathbf{t}=t_x\mathbf{e}_x+t_y\mathbf{e}_y.$$

Equations (3) and (4) can be written in the following forms

$$w(x,y,z)=w_0(z)-\frac{\partial \mathbf{U}}{\partial z}\cdot\mathbf{R}, \tag{5}$$

$$T(x,y)=t_0+\mathbf{t}\cdot\mathbf{R}. \tag{6}$$

From equations (1), (2) and (5) it follows that

$$\varepsilon_x=\varepsilon_y=\gamma_{xy}=\gamma_{xz}=\gamma_{zy}=0, \tag{7}$$

$$\varepsilon_z=\frac{\partial w_0}{\partial z}-\frac{\partial^2\mathbf{U}}{\partial z^2}\cdot\mathbf{R}. \tag{8}$$

In equations (7) and (8) ε_x, ε_y, ε_z are the normal strains, while γ_x, γ_y, γ_z are the shearing strains. The dot between two vectors denotes the scalar product in equations (5), (6) and (8). Application of the one–dimensional version of the Duhamel–Neumann law [5] gives the following result for the axial normal stress σ_z

$$\sigma_z=E(\varepsilon_z-\alpha T)=E\left(\frac{\partial w_0}{\partial z}-\frac{\partial^2\mathbf{U}}{\partial z^2}\cdot\mathbf{R}-\alpha t_0-\alpha\mathbf{t}\cdot\mathbf{R}\right), \tag{9}$$

where α is the coefficient of linear thermal expansion which does not depend on the temperature field in the considered interval of temperature change given by equation (9).

It should be noted that the displacement field formulated in equation (5) satisfies the prescriptions of the Euler–Bernoulli beam theory all shearing strains and normal strains ε_x, ε_y are zero.

2. DETERMINATION OF THE DISPLACEMENT FIELD

At first the expressions of applied axial force $\mathbf{F}=N\mathbf{e}_z$ and bending moment $M=M_x\mathbf{e}_x+M_y\mathbf{e}_y$ will be derived. It is obvious, that

$$N=\int_{(A)}\sigma_z\mathrm{dA}=EA\left(\frac{\partial w_0}{\partial z}-\alpha_0 T\right), \tag{10}$$

in which *E* denotes the Young modulus of the linearly elastic material, furthermore

$$\mathbf{e}_z\times\mathbf{M}=\int_{(A)}\mathbf{e}_z\times(\mathbf{R}\times\mathbf{e}_z\sigma_z)\mathrm{dA}=\int_{(A)}(\mathbf{R}\sigma_z)\mathrm{dA}=-E\mathbf{I}\cdot\frac{\partial^2\mathbf{U}}{\partial z^2}-\alpha E\mathbf{I}\cdot\mathbf{t}, \tag{11}$$

where **I** is the Euler tensor of the cross section A, which is defined as

$$\mathbf{I}=I_x\mathbf{e}_x\circ\mathbf{e}_x+I_{xy}(\mathbf{e}_x\circ\mathbf{e}_y+\mathbf{e}_y\circ\mathbf{e}_x)+I_y\mathbf{e}_y\circ\mathbf{e}_y, \tag{12}$$

$$I_x=\int_{(A)}x^2\mathrm{dA},\ I_{xy}=I_{xy}=\int_{(A)}yx\mathrm{dA},\ I_y=\int_{(A)}y^2\mathrm{dA}, \tag{13}$$

Here, it was used that $\int_A\mathbf{R}\mathrm{d}A=\mathbf{0}$. In equation (11) the cross between two vectors denotes their vector product and in equation (12) the dyadic product of two vectors is indicated by a circle. It is assumed that, the displacements satisfy the following homogeneous intermediate boundary conditions at $z=0$

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$$w_0(0)=0,$$
$$\mathbf{U}(0)=\mathbf{0}, \qquad (14)_1$$
$$\left.\frac{\partial \mathbf{U}}{\partial z}\right|_{z=0}=\mathbf{0}.$$

Integration of equations (10) and (11) under the conditions (14) gives

$$w_0(z)=\left(\frac{N}{AE}+\alpha t_0\right)z, \qquad -L\le z\le L, \tag{15}$$

$$\mathbf{U}(z)=\left(\frac{1}{E}\mathbf{I}^{-1}\cdot\left(\mathbf{M}\times\mathbf{e}_z\right)+\alpha\mathbf{t}\right)\frac{z^2}{2}, \qquad -L\le z\le L. \tag{16}$$

In equation (16) $\mathbf{I}^{-1}$ is the inverse of the Euler tensor, that

$$\mathbf{I}^{-1}\cdot\mathbf{I}=\mathbf{I}\cdot\mathbf{I}^{-1}=\mathbf{E}, \tag{17}$$

where $\mathbf{E}$ is the second order two–dimensional unit tensor.

3. COMPENSATION OF THE DEFORMATION CAUSED BY THE THERMAL LOAD WITH MECHANICAL LOADS

From equation (16) it follows, that

$$w_0(z)=0, \qquad -L\le z\le L, \tag{18}$$

if

$N=-\alpha AEt_0$. (19)

It is easy to show that

$$\mathbf{U}(z)=0, \qquad -L\le z\le L. \tag{20}$$

$$\mathbf{M}=\alpha E\mathbf{I}\cdot\left(\mathbf{t}\times\mathbf{e}_z\right). \tag{21}$$

4. NUMERICAL EXAMPLE

This example deals with the determination of normal force and bending moment vector which compensate the deformation caused by the thermal loading. The following numerical data are used

$$\alpha=1.2\cdot10^{-5}\frac{1}{\mathrm{K}},\ E=2\cdot10^{11}\,\mathrm{Pa},\ t_0=600\ \mathrm{K},\ t_x=4000\frac{\mathrm{K}}{\mathrm{m}},\ t_y=8000\frac{\mathrm{K}}{\mathrm{m}}.$$

A rectangular cross section is considered whose sides are

$$a=0.02\ \mathrm{m},\ b=0.04\ \mathrm{m}\ \text{ and }\ A=ab,\ I_x=\frac{a^3b}{12},\ I_y=\frac{ab^3}{12},\ I_{xy}=0.$$

A detailed computation gives the following numerical results:

$$N=-\alpha EAt_0=-1.2096\cdot10^5\,\mathrm{N}, \tag{22}$$

$$M_x=\alpha E\frac{a^3b}{12}t_y=537.6\mathrm{Nm}, \tag{23}$$

$$M_y=-\alpha E\frac{a^3b}{12}t_x=-1075.2\mathrm{Nm}. \tag{24}$$

5. CONCLUSIONS

The determination of the deformation of homogeneous isotropic linearly elastic beam under the action of axial loading and thermal loading is presented. The applied mechanical load is an axial force and bending moment. The thermal loading is obtained from the temperature field which is independent of the axial coordinate and a given linear function of the cross– sectional coordinates. The kinematics of the deformation of the beam satisfies the requirements of the Euler–Bernoulli's beam theory. The deformation caused by the thermal load can be eliminated by suitably chosen mechanical loading. The equations presented in this paper can be used as Benchmark solutions to verify other numerical methods.

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