[1.] Ravindra PAKADE, [2.] Vinod VARGHESE, [3.] Sonal BHOYAR

A SIMPLIFIED APPROACH FOR THE THERMOELASTIC ANALYSIS OF A SEMI-INFINITE ELLIPTICAL CYLINDER USING GREEN'S FUNCTION

[1.] Department of Mathematics, Priyadarshini College of Engineering, Nagpur, INDIA
[2.] Department of Mathematics, SMT Sushilabai Rajkamalji Bharti Science College, Arni, Yavatmal, INDIA
[3.] Department of Mathematics, Mahatma Gandhi Science College, Armori, Gadchiroli, INDIA

Abstract: In this paper deals with the axisymmetric temperature distribution of an elliptical cylinder subjected to the interior heat supply within which sources are generated in step with a linear function of the temperature. The solution of the heat conduction equation and the corresponding initial and boundary conditions are obtained in an analytical kind by using the Green's function methodology. The outcomes are obtained in a series in terms of Mathieu functions. The solution is more verified by comparing them to the limiting case of a circle as a special kind of ellipse. Finally, the governing equation for thermally induced deflection results and its associated stresses are found throughout the bending of a simply supported cylinder. The numerical results obtained using these computational tools and illustrated graphically.

Keywords: Elliptical cylinder, temperature distribution, thermal stresses, Green's function, thermal bending moment

1. INTRODUCTION

As far as theoretical mechanics is concerned, the solution methods for nonlinear differential equation plays a very crucial role as many problems have been modelled using such equations. In particular, bending analysis for any structural profile subjected to pressure load or thermal impact can be described by the same nonlinear differential equation. The problem of thermoelastic bending and its associated stresses has attracted much attention, and few different theories like higher-order shear and normal deformation theory [1], trigonometric shear deformation theory [2], four variable refined plate theory [3] have been suggested to solve it.

Due to the complexity of the thermoelasticity problems, mostly analytic solutions were preferred for axisymmetrical problems and other simple problems, whereas for general non-axisymmetrical problems the numerical computation was adopted as the main method. For bending problems using the Green function for thermal elastic problems has been investigated by many researchers due to several advantages [like (i) it is a powerful and flexible method (ii) it has a systematic procedure which is easily available, and so forth]. Though there are several methods outlined for finding Green's functions associated with any partial differential equation for arbitrary domains, including the method of images, separation of variables, and Laplace transforms [4].

Even few other highly cited literature reviews with Green's functions approaches which were mostly used to obtain heat conduction solutions was either in classical position [5, 6, 7] or new ones such as [8, 9, 10, 11] in their books for many decades. Of most recent literature, some authors have undertaken the work on bending analysis, which can be summarised as given below. Kim and Noda [12] derived Green's functions for solving the deflection and the transient temperature using the Galerkin method and the laminate theory. Feng and Michaelides [13] introduced a novel method named as modified Green's functions (MGFs) to obtain the transient temperature field in a homogeneous or a composite solid body. Lu et al. [14] presented an analytical method leading to the solution of transient temperature field in the multidimensional composite circular cylinder using separation of variables. Kidawa-Kukla [15, 16] proposed the heat conduction solutions for circular as well as for thin annular plate subjected to moving heat source which changes its place with time along a concentric circular trajectory be obtained by using the Green's function.

Recently, Kukla and Siedlecka [17] derived from the Green's function using eigenproblem method for the heat conduction problems in a finite multi-layered hollow cylinder. Similarly, very recently, Bialecki and Buliński [18] further investigated by a generalisation of Green's idea by obtaining heat conduction solution using Green's function technique (GFT). However, it was observed from the previous literature that almost all researcher has considered point instantaneous heat source either as a function of Dirac Delta or Heaviside. Things get further complicated when internal heat generation persists on the object under

ISSN 1584 – 2665 (printed version); ISSN 2601 – 2332 (online); ISSN-L 1584 – 2665

consideration is according to the linear function of the temperature, and further becomes unpredictable when sectional heat supply is impacted on the body. Unfortunately, we cannot comprehend each differential equation, and almost all majority phenomena are governed by nonlinear differential equations, of which most aren't tractable [19]. Thus, both analytical and numerical procedures have turned out to be the best technique to take care of such problems. In any case, numerical solutions are favoured and prevalent in practice, due to either non-accessibility or mathematical complexity of the corresponding exact solutions. Rather, limited utilisation of analytical solutions should mustn't diminish their merit over numerical ones; since exact solutions, if available, provide an insight into the governing physics of the problem, that is often missing in any numerical solution [20]. One such problem is to determine the exact temperature distribution and thermoelastic behaviour in elliptical coordinates with objects subjected to the internal heat sources which are according to the linear function of the temperature in mediums. It was learnt that there are numerous applications involving elliptical geometry require an evaluation of temperature distribution and its thermal effect on it. One distinctive example is an elliptical nuclear fuel rod in nuclear reactors.

However, to the best of authors' knowledge, no work has concerned with the thermoelastic bending analysis taking into consideration of thermal moments and resultant forces in the strain energy equation are investigated. Owing to this gap of research in this field, the authors have been motivated to conduct this study. In this present paper, the realistic problem of a thin elliptical cylinder subjected to arbitrary initial temperature on the upper lower face, with simply supported thermally insulated are studied using a typical superposition technique. This manuscript intends to present and illustrate a unified solution method, namely eigenvalue and the method of the integral transform for the Green's function derived for the differential equation with a nonhomogeneous term of a point source. The thermal moment is derived on the basis of temperature field, whereas maximum normal stresses are derived based on resultant bending moments per unit width [21, 22]. The theoretical calculation has been considered using the dimensional parameter, whereas, graphical calculations are carried out using the dimensionless parameter. The success of this novel research mainly lies on the new mathematical procedures which present a much simpler approach for optimisation of the design in terms of material usage and performance in engineering problem, particularly in the determination of thermoelastic behaviour in elliptical cylinder engaged as the pressure vessels, furnaces, etc. Finally, by considering a circle as a special kind of ellipse, it is seen that the temperature distribution in a circular solution can be derived as a special case from the present mathematical solution.

2. FORMULATION OF THE PROBLEM

Consider an elliptical cylinder occupying the space $D = \{(\xi,\eta,z) \in R^3 : \xi_0 < \xi < \infty,\ 0 < \eta < 2\pi,\ -1 \le z \le 1\}$ with length 2c is the distance between their common foci, which can be defined as $2c = 2\sqrt{a^2 - b^2}$. Now, we assume that the elliptical cylinder along the inner semi-major axis *a*, whereas inner semi-minor axis *b*. The parameter ξ defines the interfocal lines having the range $\xi \in (\xi_0, \infty)$, and that can be given as $\xi_0 = \tanh^{-1}(b/a)$.

Temperature distribution analysis

The governing equation of temperature distribution is given as follows

$$h^2\left(\frac{\partial^2 T}{\partial \xi^2} + \frac{\partial^2 T}{\partial \eta^2}\right) + \frac{\partial^2 T}{\partial z^2} + \frac{\wp(\xi,\eta,z,t,T)}{\lambda} = \frac{1}{\kappa}\frac{\partial T}{\partial t} \tag{1}$$

subjected to the initial and boundary conditions

$$\left.\begin{aligned} T(\xi,\eta,z,t)\big|_{t=0} = 0, \quad & \frac{\partial}{\partial \xi}T(\xi,\eta,z,t)\Big|_{\xi=\xi_0} = 0, \\ \lambda\frac{\partial}{\partial z}T(\xi,\eta,z,t)\Big|_{z=1} &= \alpha_0[T_0 - T(a,\eta,1,t)], \\ \lambda\frac{\partial}{\partial z}T(\xi,\eta,z,t)\Big|_{z=-1} &= -\alpha_0[T_0 - T(a,\eta,-1,t)] \end{aligned}\right\} \tag{2}$$

in which $\wp(\xi,\eta,z,t,T)$ is the volumetric energy generation, thermal diffusivity as $\kappa = \lambda/\rho C_v$, α_0 is the heat transfer coefficient, T_0 is the observed temperature of the surrounding medium and $h^2 = 2/[c^2(\cosh 2\xi - \cos 2\eta)]$.

ISSN 1584 – 2665 (printed version); ISSN 2601 – 2332 (online); ISSN-L 1584 – 2665

Now we assume that $\wp(\xi,\eta,z,t,T)$ as

$$\left.\begin{aligned}&\wp(\xi,\eta,z,t,T)=\Phi(\xi,\eta,z,t)+\psi(t)\,\theta(\xi,\eta,z,t),\\&\theta(\xi,\eta,z,t)=T(\xi,\eta,z,t)\,\exp[-\textstyle\int_0^t\psi(\tau)\,d\tau],\\&\chi(\xi,\eta,z,t)=\Phi(\xi,\eta,z,t)\exp[-\textstyle\int_0^t\psi(\tau)\,d\tau]\end{aligned}\right\}\tag{3}$$

in which $\Phi(\xi,\eta,z,t)$ is a function of coordinates and the time, but $\psi(t)$ is a function of the time only. Substituting Equation (3) in the heat conduction Equation (1) and boundary conditions (2), we assume the equivalent form as

$$h^2\left(\frac{\partial^2\theta}{\partial\xi^2}+\frac{\partial^2\theta}{\partial\eta^2}\right)+\frac{\partial^2\theta}{\partial z^2}+\frac{\chi(\xi,\eta,z,t)}{\lambda}=\frac{1}{\kappa}\frac{\partial\theta}{\partial t}\tag{4}$$

$$\left.\begin{aligned}&\theta(\xi,\eta,z,t)\big|_{t=0}=0,\quad \frac{\partial}{\partial\xi}\theta(\xi,\eta,z,t)\Big|_{\xi=\xi_0}=0,\\&\lambda\frac{\partial}{\partial z}\theta(\xi,\eta,z,t)\Big|_{z=1}=\alpha_0[\theta_0-\theta(a,\eta,1,t)],\\&\lambda\frac{\partial}{\partial z}\theta(\xi,\eta,z,t)\Big|_{z=-1}=-\alpha_0[\theta_0-\theta(a,\eta,-1,t)]\end{aligned}\right\}\tag{5}$$

In this study, it is assumed for the sake of brevity that the thermal energy is provided on the cylinder surface has the form

$$\chi(\xi,\eta,z,t)=\hat\theta\,\delta(\xi-\xi_0,\eta-\eta_0)\,\delta(z-z_0)\,\delta(t-t_0)\tag{6}$$

in which $\xi_0<\xi<\infty,\ 0<\eta_0<2\pi,\ -1<z_0<1,\ 0<t,\ t_0$, $\hat\theta$ characterises the stream of the heat and $\delta(\ )$ is the Dirac delta function.

Thermal deflections formulation

The result of the above heat conduction gives resultant moment as

$$M_T=\alpha E\int_{-1}^{1}zT\,dz\tag{7}$$

in which α and E denoting coefficient of linear thermal expansion and Young's Modulus of the material of the cylinder respectively.

The differential equation for normal deflection of the cylinder is given as

$$D\nabla^4\omega=-\frac{\nabla^2M_T}{1-\upsilon}\tag{8}$$

in which ∇^2 denotes the two-dimensional Laplacian operator in $(\xi,\ \eta)$ [24], υ denotes the Poisson's ratio and *D* is the flexural rigidity of the cylinder given as $D=Eh^3/[12(1-\upsilon^2)]$.

The boundary condition for simply supported cylinder is taken as

$$\omega\big|_{\xi=\xi_0}=\frac{\partial\omega}{\partial\xi}\Big|_{\xi=\xi_0}=0\tag{9}$$

Thermal bending stresses

The maximum normal stresses acting on those sections are parallel to ξz or ηz planes. Furthermore, the thermal stress components can be determined using small deflection and resultant moment as

$$\begin{aligned}\sigma_{\xi\xi}&=\frac{6}{\ell^2}\left\{Dh^2\left[\left(\frac{\partial^2\omega}{\partial\xi^2}+\upsilon\frac{\partial^2\omega}{\partial\eta^2}\right)-\frac{(1-\upsilon)\sinh2\xi}{(\cosh2\xi-\cos2\eta)}\frac{\partial\omega}{\partial\xi}+\frac{(1-\upsilon)\sin2\eta}{(\cosh2\xi-\cos2\eta)}\frac{\partial\omega}{\partial\eta}\right]+\frac{M_T}{1-\upsilon}\right\}\\\sigma_{\eta\eta}&=\frac{6}{\ell^2}\left\{Dh^2\left[\left(\upsilon\frac{\partial^2\omega}{\partial\xi^2}+\frac{\partial^2\omega}{\partial\eta^2}\right)+\frac{(1-\upsilon)\sinh2\xi}{(\cosh2\xi-\cos2\eta)}\frac{\partial\omega}{\partial\xi}-\frac{(1-\upsilon)\sin2\eta}{(\cosh2\xi-\cos2\eta)}\frac{\partial\omega}{\partial\eta}\right]+\frac{M_T}{1-\upsilon}\right\}\\\sigma_{\xi\eta}&=\frac{6}{\ell^2}\left\{Dh^2\left[\frac{\partial\omega}{\partial\xi}\sin2\eta+\frac{\partial\omega}{\partial\eta}\sinh2\xi-\frac{\partial^2\omega}{\partial\xi\partial\eta}(\cosh2\xi-\cos2\eta)\right]\right\}\end{aligned}\tag{10}$$

The equations (1) to (10) constitute the mathematical formulation of the problem under consideration.

ISSN 1584 – 2665 (printed version); ISSN 2601 – 2332 (online); ISSN-L 1584 – 2665

3. SOLUTION TO THE PROBLEM

Temperature distribution

The solution to the governing equation which is in an analytical form is derived by using the properties of the Green's function approach. The Green's function for the heat conduction problem describes the temperature distribution induced by the temporary, local energy impulse. The Green's function is a solution to the differential equation (4) as

$$h^2\left(\frac{\partial^2 G}{\partial \xi^2}+\frac{\partial^2 G}{\partial \eta^2}\right)+\frac{\partial^2 G}{\partial z^2}-\frac{1}{\kappa}\frac{\partial G}{\partial t}=\frac{\delta(\xi-\xi',\eta-\eta')\delta(z-\zeta)\delta(t-\tau)}{\lambda} \tag{11}$$

in which $\xi_0 \le \xi' \le \infty,\ 0<\eta'<2\pi$ and $-1\le z' \le 1$.

Moreover, the Green's function also satisfies the initial and homogeneous boundary conditions analogous to conditions (2) as

$$\left.\begin{aligned} &G(\xi,\eta,z,t)\big|_{t=0}=0,\ \frac{\partial}{\partial \xi}G(\xi,\eta,z,t)\Big|_{\xi=\xi_0}=0,\\ &\frac{\partial}{\partial z}G(\xi,\eta,z,t)\Big|_{z=1}=\mu_o G(\xi,\eta,1,t),\\ &\frac{\partial}{\partial z}G(\xi,\eta,z,t)\Big|_{z=-1}=-\mu_o G(\xi,\eta,-1,t) \end{aligned}\right\} \tag{12}$$

where $\mu_o=\alpha_o/\xi_0$.

In order to solve equation (11) under the boundary condition (12), we firstly assume the solutions in the form of a series as

$$G(\xi,\eta,z,t)=\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\sum_{p=1}^{\infty}\Theta_{2m,n}(\xi,\eta,t)\,\psi_p(z) \tag{13}$$

Substituting equation (13) in the differential equation (11) and the boundary conditions (12), yields the eigenvalue

$$\frac{\partial^2 \psi_p}{\partial z^2}+\beta^2\psi_p=0 \tag{14}$$

and the boundary conditions along z-directional as

$$\left.\frac{d\psi_p}{\partial z}+\mu_0\psi_p\right|_{z=1}=0,\ \left.\frac{d\psi_p}{\partial z}-\mu_0\psi_p\right|_{z=-1}=0 \tag{15}$$

in which $\Psi_p(z)$ are the eigenfunction corresponding to the eigenvalue.

The expression $\Psi_p(z)$ can be expressed as [25]

$$\psi_p(z)=\beta_p\cos(\beta_p z)+\mu_0\sin(\beta_p z),\qquad p=1,2,... \tag{16}$$

in which eigenvalues β_p are the roots of the positive roots of the characteristic equation $2\mu_0\beta_p\cos\beta_p-(\beta_p^2-\mu_0^2)\sin\beta_p=0$.

These functions are pairwise orthogonal so that the following conditions are satisfied

$$\int_{-1}^{1}\psi_p(z)\psi_q(z)\,dz=\begin{cases}0 & \text{for } p\ne q\\ Q_p & \text{for } p=q\end{cases} \tag{17}$$

in which

$$Q_p=\int_{-1}^{1}\psi_p^2(z)dz=(\beta_p^2+\mu_0^2)+\frac{(\beta_p^2-\mu_0^2)\cos\beta_p\sin\beta_p}{\beta_p} \tag{18}$$

Now, the Dirac function $\delta(z-\zeta)$ in equation (11) can be re-expressed in the form

$$\delta(z-\zeta)=\sum_{p=1}^{\infty}\frac{\psi_p(z)\psi_p(\zeta)}{Q_p} \tag{19}$$

Substituting equation (13) and (19) into equation (11) gives

$$h^2\left[\left(\frac{\partial^2}{\partial \xi^2}+\frac{\partial^2}{\partial \eta^2}\right)-\beta_p^2\right]\Theta_{2m,n}-\frac{1}{\kappa}\frac{\partial \Theta_{2m,n}}{\partial t}=\frac{\delta(\xi-\xi',\eta-\eta')\delta(t-\tau)\psi_p(\zeta)}{\lambda Q_p} \tag{20}$$

and the boundary condition (12) is rewritten as

ISSN 1584 – 2665 (printed version); ISSN 2601 – 2332 (online); ISSN-L 1584 – 2665

$$\Theta_{2m,n}(\xi,\eta,t)\big|_{t=0}=0,\quad \frac{\partial}{\partial\xi}\Theta_{2m,n}(\xi,\eta,t)\Big|_{\xi=\xi_0}=0 \tag{21}$$

Now in order to derive the eigenfunctions $\Theta_{2m,n}(\xi,\eta,t)$ from the differential equation (20), we first introduce the integral transform [26] of order n and m over the variable ξ and η as

$$\bar{f}(q_{2n,m})=\int_{\xi_0}^{\infty}\int_0^{2\pi} f(\xi,\eta)\aleph_{2n,m}(\xi,\eta)(\cosh 2\xi-\cos 2\eta)\,d\xi\,d\eta \tag{22}$$

in which $q_{2n,m}$ is the root of the transcendental equation $Ce_{2n}(\xi,q_{2n,m})=0$, $ce_n(\eta,q)$ [24, pp.21] is a Mathieu function of the first kind of order n, $Ce_n(\xi,q)$ [24, pp.27] is a modified Mathieu function of the first kind of order n. The inversion formula was given [26] as

$$f(\xi,\eta)=\frac{1}{\pi}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}(-1)^m\int_{\xi_0}^{\infty}C_{2n}^2\bar{f}(q_{2n,m})\aleph_{2n,m}(\xi,\eta)\,d\xi \tag{23}$$

where $\aleph_{2n,m}(\xi,\eta)=\dfrac{Ce_{2n}(\xi,q_{2n,m})Fey_{2n}(\xi_0,q_{2n,m})-Fey_{2n}(\xi,q_{2n,m})Ce_{2n}(\xi_0,q_{2n,m})}{\sqrt{[Ce_{2n}(\xi_0,q_{2n,m})]^2+[Fey_{2n}(\xi_0,q_{2n,m})]^2}}ce_{2n}(\eta,q_{2n,m})$

and $\bar{f}(q_{2n,m})=\int_{\xi_0}^{\infty}\int_0^{2\pi} f(\xi,\eta)\Xi_{2n,m}(\xi,\eta)(\cosh 2\xi-\cos 2\eta)\,d\xi\,d\eta$

$$\Xi_{2n,m}(\xi,\eta)=\frac{Ce_{2n}(\xi,q_{2n,m})Fey'_{2n}(\xi_0,q_{2n,m})-Fey_{2n}(\xi,q_{2n,m})Ce'_{2n}(\xi_0,q_{2n,m})}{\sqrt{[Ce'_{2n}(\xi_0,q_{2n,m})]^2+[Fey'_{2n}(\xi_0,q_{2n,m})]^2}}ce_{2n}(\eta,q_{2n,m})$$

We find that $\overline{\Theta}_{2m,n}$ which satisfies the differential equation

$$\frac{\partial}{\partial t}\overline{\Theta}_{2m,n}+\kappa(\alpha_{2m,n}^2+\beta_p^2)\overline{\Theta}_{2m,n}=-\kappa\left(\frac{\psi_p(\zeta)}{\lambda Q_p}\right)\exp(-\alpha_{2m,n}^2\xi')\exp(-\alpha_{2m,n}^2\eta')\delta(t-\tau) \tag{24}$$

with the initial condition

$$\overline{\Theta}_{2m,n}(q_{2m,n},t)\big|_{t=0}=0 \tag{25}$$

in which $\overline{\Theta}_{2m,n}$ is the transformed function of $\Theta_{2m,n}$ and $\alpha_{2m,n}^2=4q_{2m,n}/c^2$ and $q_{2m,n}$ are the root of the transcendental equation $Ce_{2n}(a,q_{2m,n})=0$.

Taking the Laplace transform of equation (24) and bearing in account the equation (25), we get mathematical simplification as

$$\hat{\overline{\Theta}}_{2m,n}=-\kappa\left(\frac{\psi_p(\zeta)}{\lambda Q_p}\right)\left[\frac{\exp(-\alpha_{2m,n}^2\xi')\exp(-\alpha_{2m,n}^2\eta')}{s+\kappa(\alpha_{2m,n}^2+\beta_p^2)}\right]\exp(-\tau s) \tag{26}$$

in which $\hat{\overline{\Theta}}_{2m,n}$ is the transformed function of $\overline{\Theta}_{2m,n}$,

moreover, the Laplace inversion formula gives

$$\overline{\Theta}_{2m,n}=-\kappa\left(\frac{\psi_p(\zeta)}{\lambda Q_p}\right)\exp(-\alpha_{2m,n}^2\xi')\exp(-\alpha_{2m,n}^2\eta')\times\int_0^t\delta(u-\tau)\exp\left[-\kappa(\alpha_{2m,n}^2+\beta_p^2)\right](t-u)\,du \tag{27}$$

then accomplishing inversion theorems of the transform rules defined by equations (23) on equation (27), yields

$$\begin{aligned}\Theta_{2m,n}=\frac{1}{\pi}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}(-1)^m\int_0^{\infty}C_{2n}^2\Bigg\{-\kappa\left(\frac{\psi_p(\zeta)}{\lambda Q_p}\right)\exp(-\alpha_{2m,n}^2\xi')\\ \times\exp(-\alpha_{2m,n}^2\eta')\int_0^t\delta(u-\tau)\exp\left[-\kappa(\alpha_{2m,n}^2+\beta_p^2)\right](t-u)\,du\Bigg\}\\ \times A_{2n}(\xi,q_{2n,m})ce_{2n}(\eta,q_{2n,m})\,d\xi\end{aligned} \tag{28}$$

Finally substituting equations (16) and (28) into equation (13), the Green function has the form

$$\begin{aligned}G(\xi,\eta,z,t\,|\,\xi',\eta',\zeta,\tau)=\frac{1}{\pi}\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\sum_{p=1}^{\infty}(-1)^m\int_0^{\infty}C_{2n}^2[\beta_p\cos(\beta_p z)+\mu_0\sin(\beta_p z)]\\ \times\left\{-\kappa\left(\frac{\psi_p(\zeta)}{\lambda Q_p}\right)\exp(-\alpha_{2m,n}^2\xi')\exp(-\alpha_{2m,n}^2\eta')\int_0^t\delta(u-\tau)\right.\\ \left.\times\exp\left[-\kappa(\alpha_{2m,n}^2+\beta_p^2)\right](t-u)\,du\right\}A_{2n}(\xi,q_{2n,m})ce_{2n}(\eta,q_{2n,m})\,d\xi\end{aligned} \tag{29}$$

The temperature distribution $\theta(\xi,\eta,z,t)$ is expressed by Green's function as

ISSN 1584 – 2665 (printed version); ISSN 2601 – 2332 (online); ISSN-L 1584 – 2665

$$\theta(\xi,\eta,z,t)=\int_0^t d\tau\int_{\xi_0}^{\infty} d\xi'\int_0^{2\pi} d\eta'\int_{-1}^{1}\chi(\xi',\eta',\zeta,\tau)\,G(\xi,\eta,z,t\,|\,\xi',\eta',\zeta,\tau)\,d\zeta \tag{30}$$

Taking into account the second equation of equation (3), the temperature distribution is finally represented by

$$T(\xi,\eta,z,t)=\exp[\textstyle\int_0^t\psi(\tau)\,d\tau]\int_0^t d\tau\int_{\xi_0}^{\infty} d\xi'\int_0^{2\pi} d\eta'\int_{-1}^{1}\chi(\xi',\eta',\zeta,\tau) \times G(\xi,\eta,z,t\,|\,\xi',\eta',\zeta,\tau)\,d\zeta \tag{31}$$

Substituting the value of equation (29) into equation (30), one yield

$$\begin{aligned}T(\xi,\eta,z,t)=&\frac{1}{\pi}\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\sum_{p=1}^{\infty}\int_0^{\infty}C_{2n}^2\left\{[\beta_p\cos(\beta_p z)+\mu_0\sin(\beta_p z)]\right.\\&\times\exp[\textstyle\int_0^t\psi(\tau)\,d\tau]\int_0^t d\tau\int_{\xi_0}^{\infty}d\xi'\int_0^{2\pi}d\eta'\int_{-1}^{1}\chi(\xi',\eta',\zeta,\tau)\\&\times\left(\frac{-\kappa\psi_p(\zeta)}{\lambda Q_p}\right)\exp(-\alpha_{2m,n}^2\,\xi')\exp(-\alpha_{2m,n}^2\,\eta')\\&\times\left.\mathrm{A}_{2n}(\xi,q_{2n,m})ce_{2n}(\eta,q_{2n,m})\right\}d\zeta\, d\xi\\&\times\int_0^t\delta(u-\tau)\exp\left[-\kappa(\alpha_{2m,n}^2+\beta_p^2)\right](t-u)\,du\end{aligned} \tag{32}$$

The above function is given in equation (32) represents the temperature at every instance and at all points of an elliptical cylinder of finite height under the influence of boundary conditions.

Thermal deflections analysis

Substituting the value of equation (32) into equation (7), one yields resultant moment

$$\begin{aligned}M_T=&\frac{\alpha E}{\pi}\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\sum_{p=1}^{\infty}\int_0^{\infty}C_{2n}^2[2\mu_0(-\beta_p\cos\beta_p)+\sin\beta_p]\\&\times\left\langle\exp[\textstyle\int_0^t\psi(\tau)\,d\tau]\int_0^t d\tau\int_{\xi_0}^{\infty}d\xi'\int_0^{2\pi}d\eta'\right.\\&\times\int_{-1}^{1}\left\{-\kappa\left(\frac{\psi_p(\zeta)}{\lambda Q_p}\right)\chi(\xi',\eta',\zeta,\tau)\exp(-\alpha_{2m,n}^2\,\xi')\right.\\&\times\left.\exp(-\alpha_{2m,n}^2\,\eta')\times\mathrm{A}_{2n}(\xi,q_{2n,m})ce_{2n}(\eta,q_{2n,m})\right\}d\zeta\, d\xi\\&\times\left.\int_0^t\delta(u-\tau)\exp\left[-\kappa(\alpha_{2m,n}^2+\beta_p^2)\right](t-u)\,du\right\rangle/\beta_p^2\end{aligned} \tag{33}$$

Now substituting the value of equation (32) into equation (8), one gets

$$\begin{aligned}\omega=&\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\sum_{p=1}^{\infty}\left\langle[A_{2n}\cos 2n\xi\sin 2m\eta+B_{2n}\mathrm{A}_{2n}(\xi,q_{2n,m})\right.\\&\times ce_{2n}(\eta,q_{2n,m})]+\frac{\alpha E}{D\pi(1-\upsilon)}\int_0^{\infty}C_{2n}^2\left\{\exp[\textstyle\int_0^t\psi(\tau)\,d\tau]\right.\\&\times\int_0^t d\tau\int_{\xi_0}^{\infty}d\xi'\int_0^{2\pi}d\eta'\int_{-1}^{1}\left\{-\kappa\left(\frac{\psi_p(\zeta)}{\lambda Q_p}\right)\chi(\xi',\eta',\zeta,\tau)\right.\\&\times\exp(-\alpha_{2m,n}^2\,\xi')\exp(-\alpha_{2m,n}^2\,\eta')\mathrm{A}_{2n}(\xi,q_{2n,m})\\&\times\left.ce_{2n}(\eta,q_{2n,m})\right\}d\zeta\, d\xi\\&\times\left.\int_0^t\delta(u-\tau)\exp\left[-\kappa(\alpha_{2m,n}^2+\beta_p^2)\right](t-u)\,du\right\rangle/\beta_p^2\end{aligned} \tag{34}$$

in which $\mathrm{A}_{2n,m}$ and $\mathrm{B}_{2n,m}$ are the constants to be determined from equation (34) and taking into account the boundary condition (9), one obtains

$$\begin{aligned}A_{2n}=&0,\\B_{2n}=&\left\langle[-\cosh 2n\xi_0\,\mathrm{A}'_{2n}(\xi_0,q_{2n,m})+\sinh 2na\,\mathrm{A}_{2n}(\xi_0,q_{2n,m})]\right.\\&\times\frac{\alpha E}{D(1-\upsilon)}\int_0^{\infty}C_{2n}^2\left\{\exp[\textstyle\int_0^t\psi(\tau)\,d\tau]\int_0^t d\tau\int_{\xi_0}^{\infty}d\xi'\int_0^{2\pi}d\eta'\right.\\&\times\int_{-1}^{1}\left\{-\kappa\left(\frac{\psi_p(\zeta)}{\lambda Q_p}\right)\chi(\xi',\eta',\zeta,\tau)\exp(-\alpha_{2m,n}^2\,\xi')\right.\\&\times\left.\exp(-\alpha_{2m,n}^2\,\eta')ce_{2n}(\eta,q_{2n,m})\right\}d\zeta\, d\xi\\&\times\left.\int_0^t\delta(u-\tau)\exp\left[-\kappa(\alpha_{2m,n}^2+\beta_p^2)\right](t-u)\,du\right\rangle\\&/\,ce_{2n}(\eta,q_{2n,m})Z_{2n}\end{aligned} \tag{36}$$

ISSN 1584 – 2665 (printed version); ISSN 2601 – 2332 (online); ISSN-L 1584 – 2665

in which $Z_{2n} = \beta_p^2 C_{2n}[\cosh 2n\xi_0 A'_{2n}(\xi_0, q_{2n,m}) + \sinh 2n\xi_0 A_{2n}(\xi_0, q_{2n,m})]$

Finally, by substituting the values of A_{2n} and B_{2n} from equation (36) into equation (34) results in the required expression for the thermal deflection.

Thermal bending stresses

The resulting equations of stresses can be obtained by substituting the resultant moment (33) and deflection equations (34) in equation (10). The equations of stresses are rather lengthy, and consequently the same have been omitted here for the sake of brevity, but have been considered during the graphical discussion using MATHEMATICA software.

3. TRANSITION TO CIRCULAR CYLINDER

When the elliptical cylinder tends to a circular cylinder of the radius a, the semi-focal $c \to 0$.Also $e \to 0$ [as $\xi \to \infty$], $\cosh 2\xi\, d\xi \to 2\cosh 2\xi \sinh 2\xi\, d\xi \to 2r dr / c^2$, $\sinh\xi \to \cosh\xi$, $h\cosh\xi \to r$ [as $h \to 0$], $\cosh\xi\, d\xi \to r\, dr$, $h\sinh\xi\, d\xi \to dr$,

Using results from [24]

$C_0 Ce_0(\xi, q_{0,m}) \to i^0 J_0(\lambda_m r)$, $C_0 Fey_0(\xi, q_{0,m}) \to i^0 Y_0(\lambda_m r)$, $ce_0(\eta, q_{0,m}) \to 1/\sqrt{2}$, $\aleph_{2m} \to 0$, $\lambda_{0,m}^2 = \alpha_{0,m}^2 / a^2 = \alpha_m^2 / a^2 = \lambda_m^2$.

Taking into account the aforesaid parameters, the temperature distribution in cylindrical coordinate is finally represented by

$$\begin{aligned} T(r,z,t) = {} & \frac{1}{\pi} \sum_{m=0}^{\infty} \sum_{p=1}^{\infty} \int_0^{\infty} C_0^2 \Big\{ [\beta_p \cos(\beta_p z) + \mu_0 \sin(\beta_p z)] \\ & \times \exp\left[\int_0^t \psi(\tau)\, d\tau\right] \int_0^t d\tau \int_a^{\infty} dr' \int_{-1}^{1} \chi(r,\zeta,\tau) \\ & \times \left(\frac{-\kappa \psi_p(\zeta)}{\lambda Q_p} \right) \exp(-\lambda_m^2 r') i^0 J_0(\lambda_m r) i^0 Y_0(\lambda_m r) 1/\sqrt{2} \Big\} d\zeta\, dr \\ & \times \int_0^t \delta(u-\tau) \exp[-\kappa(\lambda_m^2 + \beta_p^2)](t-u)\, du \end{aligned} \tag{37}$$

The aforementioned degenerated result agrees with the previous result [15, 16].

4. SPECIAL CASE AND NUMERICAL CALCULATIONS

For the sake of simplicity of calculation, we introduce the following dimensionless values

$$\left. \begin{aligned} & \bar{\xi} = \xi/\xi_0,\ \bar{z} = [z - (-\ell)]/\xi_0,\ e = c/\xi_0,\ \tau = \kappa t/\xi_0^2, \\ & \bar{\theta} = \theta/\theta_0, \bar{\varpi} = \omega/\alpha\theta_0\xi_0, \bar{\sigma}_{ij} = \sigma_{ij}/E\alpha\theta_0\ \ (i,j = \xi,\eta) \end{aligned} \right\} \tag{38}$$

Substituting the value of equation (38) in temperature equation (32), deflection equation (34) and components of stresses, we obtained the expressions for the temperature, deflection and thermal stresses respectively for our numerical discussion. The numerical calculation have been carried out for Aluminum metal with physical parameter ξ_0 = 1 m, ℓ = 1 m, Modulus of Elasticity E = 70 GPa, Poisson's ratio υ = 0.35, Thermal expansion coefficient, α = 23× 10^{-6} /$^{\circ}$C, Thermal diffusivity κ = 84.18 ×10^{-6} m^2/s^{-1}, Thermal conductivity λ = 204.2 Wm^{-1} K^{-1} with $q_{2n,m}$ = 0.0986, 0.3947, 0.8882, 1.5791, 2.4674, 3.5530, 4.8361, 6.3165, 7.9943, 9.8696, 11.9422, 14.2122, 16.6796, 19.3444, 22.2066, 25.2661, 28.5231, 31.9775, 35.6292 are the positive & real roots. In order to examine the influence of heating on the cylinder, the numerical calculation for all variables was performed. Numerical calculations are depicted in the following figures using MATHEMATICA software.

Figs. 2-4 illustrates the numerical results of temperature distribution, deflection and thermal stresses on elliptical cylinder due to internal heat generation within the solid under thermal boundary condition that are subjected to known temperature at any instant time τ.

Figs. 2(a) shows that the temperature distribution along the radial direction for different values of time τ and along time τ for various values of $\bar{z}$ at any instant which maximise its magnitude towards outer edge due to internal heat supply. Figure 2(b) depicts that the temperature distribution along $\bar{z}$ - direction for a different time τ, in which it is observed that the maximum temperature distribution occurs at outer core of the cylinder. Figure 2(c) illustrates temperature distribution along η - direction. The temperature distribution approach to zero at both extreme ends due to the more compressive force acting along the edges, whereas temperature attains the maximum value at the centre due to tensile force along the mid part for different values of τ following a bell-shaped temperature distribution curve.

ISSN 1584 – 2665 (printed version); ISSN 2601 – 2332 (online); ISSN-L 1584 – 2665

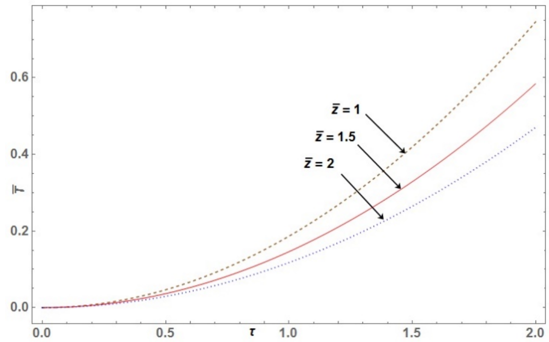

Figure 2(a): Temperature distribution along τ - direction for various value of $\bar{z}$.

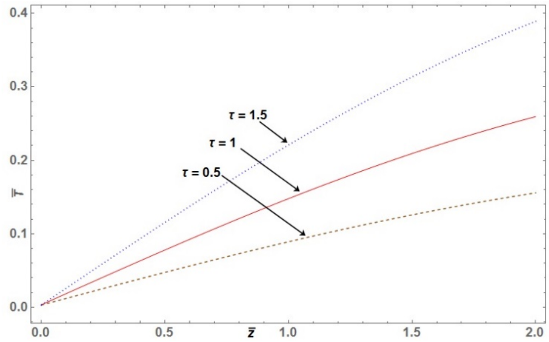

Figure 2(b): Temperature distribution along $\bar{z}$ - direction for different value of τ.

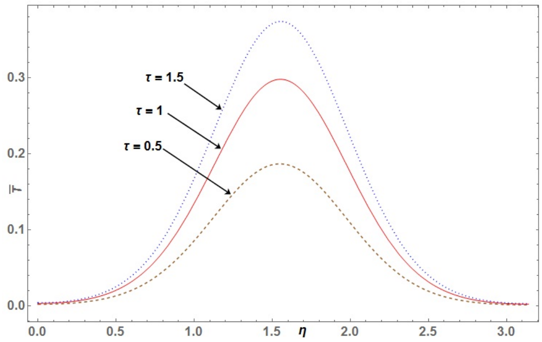

Figure 2(c): Temperature distribution along η - direction for various value of τ.

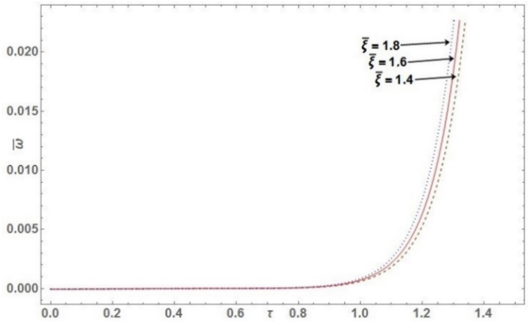

Figure 3(a): Thermal deflection along time τ for different value of $\bar{\xi}$.

In Figure 3(a) as expected gradual increase in thermal deflection along time parameter for various value of $\bar{\xi}$ was observed.

Figs. 4(a)-4(d) illustrates the thermal stresses $\bar{\sigma}_{\xi\xi}$, $\bar{\sigma}_{\eta\eta}$ and $\bar{\sigma}_{\xi\eta}$ for different value of $\bar{\xi}$, initially the stresses attain zero, and maximum stresses occurs as we move towards the time parameter τ. Figure 4illustrates the radial stress distribution $\bar{\sigma}_{\xi\xi}$ along the angular direction for a different time; it is evident from the figure that at the early stage, radial stress increases gradually and maximum radial stress occurs at the mid-core and suddenly attains minimum value, thus curves follow a dome shape. Similar curve nature was observed for $\bar{\sigma}_{\eta\eta}$ and $\bar{\sigma}_{\xi\eta}$ along the angular direction for different time τ and has been omitted here for the sake of brevity.

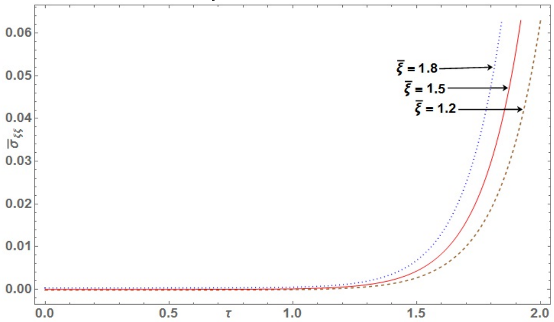

Figure 4(a): Thermal stress $\bar{\sigma}_{\xi\xi}$ along τ for different value of $\bar{\xi}$.

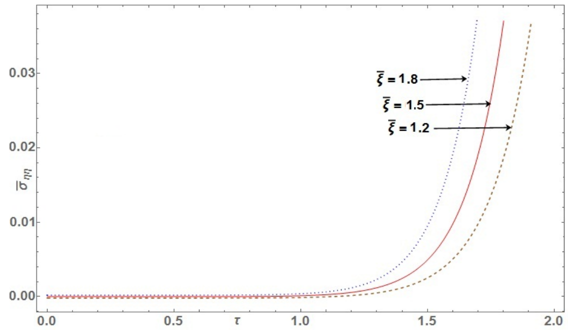

Figure 4(b): Thermal stress $\bar{\sigma}_{\eta\eta}$ along τ for various value of $\bar{\xi}$.

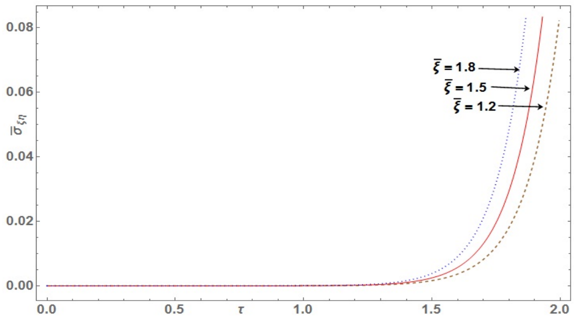

Figure 4(c): Thermal stress $\bar{\sigma}_{\xi\eta}$ along τ for different value of $\bar{\xi}$.

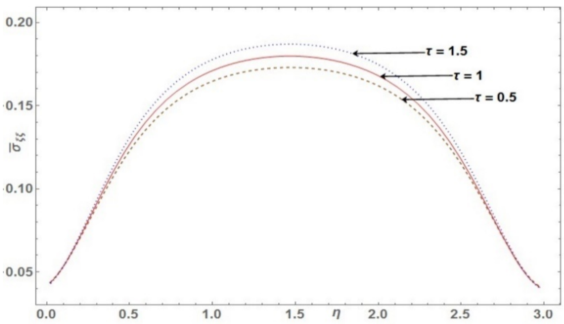

Figure 4(d): Radial stress $\bar{\sigma}_{\xi\xi}$ along η - direction for different τ.

ISSN 1584 – 2665 (printed version); ISSN 2601 – 2332 (online); ISSN-L 1584 – 2665

5. CONCLUSION

The proposed analytical solution of transient thermal stress problem of an elliptical cylinder was handled in elliptical coordinate system with the presence of a source of internal heat. To the author's knowledge, there have been no reports of solution so far in which sources are generated according to the linear function of the temperature in mediums in the form of elliptical cylinder of finite height. The analysis of non-stationary three-dimensional equation of heat conduction is investigated with the unified solution method, namely eigenvalue and the method of the integral transform for the Green's function.

References

[1] M. S. A. Houari, A. Tounsia, O. A. Bég, Thermoelastic bending analysis of functionally graded sandwich plates using a new higher order shear and normal deformation theory, International Journal of Mechanical Sciences, vol. 76, pp. 102-111(2013).
[2] A. Tounsi, M. S. A. Houari, S. Benyoucef, .E. A. A. Bedia, A refined trigonometric shear deformation theory for thermoelastic bending of functionally graded sandwich plates, Aerospace Science and Technology, vol. 24, no. 1, pp. 209-220(2013).
[3] A. S. Sayyad, Y. M. Ghugal, B. A. Mhaske, A Four-Variable Plate Theory for Thermoelastic Bending Analysis of Laminated Composite Plates, Journal of Thermal Stresses, vol. 38, no. 8, pp. 904-925(2015).
[4] K. D. Cole, J.V. Beck, A. Haji-Sheikh, B. Litkouhi, Heat conduction using Green's functions. Taylor & Francis, New York (2011).
[5] P. M. Morse, H. Fesbach, Methods of theoretical physics, Mc-Graw-Hill, New YorkMATH (1953)
[6] H.S. Carslaw, J. C. Jaeger, Conduction of heat in solids, Oxford University Press, New York (1959).
[7] G. Barton, Elements of Green's functions and propagation, Oxford University Press, LondonMATH (1989).
[8] A. G. Butkovsky, Green's functions and transfer functions handbook. Halsted Press, New York (1982).
[9] G. F. Roach, Green's functions introductory theory with applications. Van Nostrand Reinhold, New YorkMATH (1970).
[10] I. Stakgold, Green's functions and boundary value problems. Wiley-Interscience, New YorkMATH (1979).
[11] M. D. Greenberg, Applications of Green's functions in science and engineering, Prentice-Hall, Englewood Cliffs. (1971).
[12] K.-S. Kim, N. Noda, A Green's function approach to the deflection of a FGM plate under transient thermal loading, Archive of Applied Mechanics, vol. 72, pp. 127–137 (2002).
[13] F.-G. Feng, E.E. Michaelides, The use of modified Green's functions in unsteady heat transfer, International Journal Heat and Mass Transfer, vol. 40, pp. 2997-3002(1997).
[14] Lu X., Tervola P., Viljanen M., Transient analytical solution to heat conduction in composite circular cylinder, International Journal Heat and Mass Transfer, vol. 49, pp. 341-348(2006).
[15] J. Kidawa-Kukla, Temperature distribution in a circular plate heated by moving heat source, Scientific Research of the Institute of Mathematics and Computer Science, vol. 1, no. 8, pp. 71-77 (2008).
[16] J. Kidawa-Kukla, Temperature distribution in an annular plate with a moving discrete heat generation source, Scientific Research of the Institute of Mathematics and Computer Science, vol. 1, no. 8, pp. 77-84 (2008).
[17] S. Kukla, U. Siedlecka, Green's function for heat conduction problems in a multi-layered hollow cylinder, vol. 13, no. 3, pp. 115-122 (2014).
[18] R. A. Bialecki, Z. Buliński, Green's Functions in Transient Heat Conduction, Encyclopedia of Thermal Stresses, Springer Netherlands, pp. 2070-2096 (2014).
[19] K. Watanabe, Integral transform techniques for Green's function, Springer Cham Heidelberg New York Dordrecht London (2014).
[20] P. Bhad, V. Varghese, L. Singh, Thermoelastic theories on elliptical profile objects: An overview and perspective, International Journal of Advances in Applied Mathematics and Mechanics, vol. 4, no. 2, pp. 12-60 (2016).
[21] P. Bhad, V. Varghese and L. Khalsa, Thermoelastic-induced vibrations on an elliptical disk with internal heat sources, Journal of Thermal Stresses, vol. 40, no. 4, pp. 502-516 (2017).
[22] T. Dhakate, V. Varghese and L. Khalsa, Integral transform approach for solving dynamic thermal vibrations in the elliptical disk, Journal of Thermal Stresses, vol. 40, no. 9, pp. 1093-1110 (2017).
[23] I. N. Sneddon, Fourier transforms, McGraw-Hill, New York (1995).
[24] N. McLachlan, Theory and Application of Mathieu functions. Clarendon Press (1951).
[25] D.G. Duffy, Green's Functions with applications-studies in advanced mathematics, Boca Raton, New York (2001).
[26] Gupta R.K., A finite transform involving Mathieu functions and its application, Proc. Net. Inst. Sc., India, Part A, vol. 30, no. 6, pp. 779-795 (1964).

ISSN 1584 – 2665 (printed version); ISSN 2601 – 2332 (online); ISSN–L 1584 – 2665