[1.] U.H. OJIMADU, [1.]M.A. USMAN, [1.] A.O. OLASUPO, [1.] O.O. OLUBANWO, [2.]M.A. AYODELE, [3.]M.A. SULAIMAN

A COMPARATIVE ANALYTICAL SOLUTION OF TELEGRAPH EQUATION

[1.]Department of Mathematical Sciences, Olabisi Onabanjo University, Ago–Iwoye, Ogun State, NIGERIA
[2.]Open and Distance Learning Centre, Olabisi Onabanjo University, Ago Iwoye, Ogun State, NIGERIA
[3.]Department of Mathematics, Lagos State University, Ojo, Lagos State, NIGERIA

Abstract: In this paper, two methods Adomian Decomposition Method (ADM) and Laplace Adomian Decomposition Method (LADM) were adopted to solve the telegraph equation. The essence of this research is to establish a relationship between the two analytical methods. It was observed that the two methods were consistent as the results obtained from the numerical examples on the two methods were the same. We also generated the telegraph equation to help provide a solid basis for the application of the telegraph equation. The telegraph equation is one of the nonlinear partial differential equation and it's application to solving practical problems were suggested for further studies.
Keywords: Adomian Decomposition Method, Laplace Adomian Decomposition Method, Telegraph equation, Resistance, Conductance, Inductance

1. INTRODUCTION

The telegraph equation, also known as Transmission Line Equation was first developed by Kirchhoff when William Thompson (later Lord Kelvin), a great 19th century mathematical physicist was instrumental in the British effort to lay the Trans–Atlantic telegraph cable. Thompson (2013) [1] certainly had done some analysis on the Telegraph equation to draw some conclusions for the work.

In this paper, our emphasis is not on the historical background of the Telegraph equation but on providing a comparative analysis of two mathematical analytical methods, which includes Adomian Decomposition Methods (ADM) and Laplace Adomian Decomposition Methods (LADM) for solving the Telegraph equation. These two methods are suitable for solving nonlinear Partial Differential Equations especially in the field of quantum mechanics and fluid mechanics. We can apply these methods to solve problems on geochemistry, optical fibres, plasma physics, meteorology and biology. However, other methods have been used to solve nonlinear partial differential equations such as the tanh–function method [2], the extended tanh–coth method [3], the function. Exxpansion method [3], vibrational iteration method [4], the homotopy perturbation method [5], the casoration formulation [6], the extended multiple Riccati equations expansion method [7] and the enhanced (G'/G) – Expansion method [8] [9] [10]. We shall x–ray the derivation of the derivation of the Telegraph equation with proper analysis of the variables and partial derivatives. The two methods (Adomian Decomposition Method (ADM) and Laplace Adomian Decomposition Method (LADM)) will be examined.

Different researchers and scholars have done work on these two methods [11] developed a two–step Laplace decomposition method for solving nonlinear partial differential equations [12] [13] focused on using laplace decomposition method to solve Klien–Gord equation [14] used a blend or Adomian Decomposition Method (ADM) and Laplace Decomposition Method to solve Klein–Gordon equation. Our focus is to find the meeting point in order to validate the efficacy of both methods under consideration. The two methods will be applied to solve the telegraph equation and conclusions will be drawn based on results obtained from the application of both methods.

2. DERIVATION OF THE TELEGRAPH EQUATION

Given an infinitesimal piece of telegraph wire of an electrical circuit which consist of a resistor of resistance Rdx and a coil of inductance Ldx. If i(x, t) is the current through the wire, the voltage across the resistor iRdx with the voltage across the coil is $\frac{\partial i}{\partial t}\mathrm{Ldx}$. Suppose u(x, t) is the voltage at position x at time t, then the change in voltage between the ends of the piece of wire is

$$du = -iRdx - \frac{\partial i}{\partial t}Ldx \quad (1)$$

Current can escape through the wire to the ground either from a resistor of conductance Gdx or from a capacitor of conductance Cdx. The amount that escapes through the resistor is uGdx

ISSN 1584 – 2665 (printed version); ISSN 2601 – 2332 (online); ISSN-L 1584 – 2665

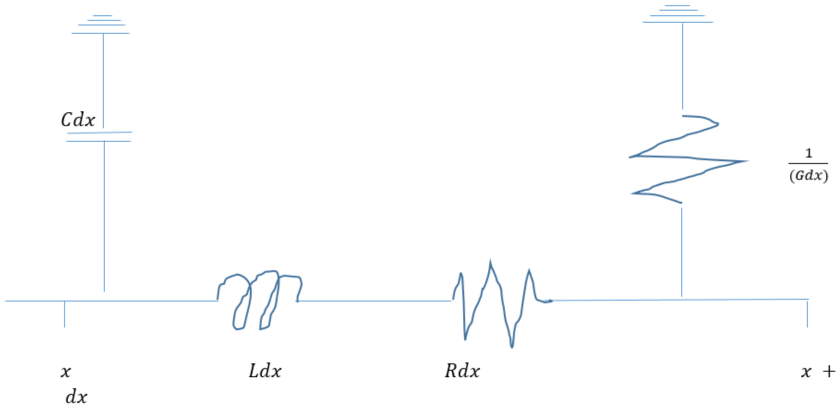

Figure 1. Electrical circuit

Let q=uCdx be the charge on the capacitor, the amount that escapes from the capacitor is $q_t = u_t Cdx$. In total;

$$di = -uGdx - u_t Cdx \tag{2}$$

Dividing equation (2) by dx

$$\frac{\partial i}{\partial x} == -uG - u_t C \tag{3}$$

Taking the limit of equation (3) as $dx \to 0$, we get the differential equation;

$$i_x + u_t C + uG = 0 \tag{4}$$

Dividing equation (1) by dx

$$\frac{du}{dx} = -iR - \frac{\partial i}{\partial x} L \tag{5}$$

Taking the limit of equation (5) as $dx \to 0$, we get the differential equation;

$$u_x + iR + i_t L = 0 \tag{6}$$

We take the partial derivative of equation (4) with respect to t;

$$i_{xt} + u_{tt} C + u_t G = 0 \tag{7}$$

Rearranging equation (7), we have

$$i_{xt} = -Cu_{tt} - Gu_t \tag{8}$$

We take the partial derivative of equation (6) with respect to x

$$u_{xx} + Ri_x + Li_{xt} = 0 \tag{9}$$

Substitute equation (4) and (8) into equation (9)

$$u_{xx} + R[-Cu_t - Gu] + L[-Cu_{tt} - Gu_t] = 0 \tag{10}$$

Divide equation (10) by LC

$$\frac{1}{LC} u_{xx} = u_{tt} + \left(\frac{R}{L} + \frac{G}{C}\right) u_t + \frac{GR}{LC} u \tag{11}$$

Renaming some constants, we get the telegraph equation

$$u_{tt} + (\alpha + \beta) u_t + \alpha\beta u = c^2 u_{xx} \tag{12}$$

where $c^2 = \frac{1}{LC}, \alpha = \frac{G}{C}, \beta = \frac{R}{L}$

3. METHOD OF SOLUTION

In providing solutions to two examples of telegraph equation. We solve these two examples using the Adomian Decomposition method and then we also apply the Laplace Adomian Decomposition Method (LADM) and then compare the results obtained in the two mathematical analytical methods.

Example 1 (Using Adomian Decomposition Method)

Solve the following homogeneous telegraph equation;

$$u_{xx} = u_{tt} + u_t - u \tag{13}$$

Subject to the conditions;

$$u(0,t) = e^{-2t}, u_x(0,t) = e^{-2x}$$
$$u(x,0) = e^x, u_t(x,0) = -2e^x \tag{14}$$

Solution

Operation with L_x^{-1} on (13) and using the boundary conditions yields

$$u(x,t) = e^{-2t} + xe^{-2t} + L_x^{-1}(u_{tt} + u_t - u) \tag{15}$$

This gives;

$$\sum_{n=0}^{\infty} u_n(x,t) = e^{-2t} + xe^{-2t} + L_x^{-1}\left(\left(\sum_{n=0}^{\infty} u_n\right)_{tt} + \left(\sum_{n=0}^{\infty} u_n\right)_t - \sum_{n=0}^{\infty} u_n\right) \tag{16}$$

ISSN 1584 – 2665 (printed version); ISSN 2601 – 2332 (online); ISSN-L 1584 – 2665

The decomposition method suggests the relation;

$$u_0(x,t) = e^{-2t} + xe^{-2t}$$

$$\vdots$$

$$u_{k+1}(x,t) = L_x^{-1}(u_{ktt} + u_{kt} - u_k), \quad K \geq 0 \tag{17}$$

When the components of the solution u(x,t) given by

$$\begin{aligned} u_0(x,t) &= e^{-2t} + xe^{-2t} \\ u_1(x,t) &= L_x^{-1}(u_{0tt} + u_{0t} - u_0) \\ &= \frac{1}{2!}x^2e^{-2t} + \frac{1}{3!}x^3e^{-2t} \\ u_2(x,t) &= L_x^{-1}(u_{1tt} + u_{1t} - u_1) \\ &= \frac{1}{4!}x^4e^{-2t} + \frac{1}{5!}x^5e^{-2t} \\ u_3(x,t) &= L_x^{-1}(u_{2tt} + u_{2t} - u_2) \\ &= \frac{1}{6!}x^6e^{-2t} + \frac{1}{7!}x^7e^{-2t} \end{aligned} \tag{18}$$

Combining the solutions, we have;

$$u(x,t) = e^{-2t}\left(1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \cdots\right),$$

Which gives the exact solution in the form

$$u(x,t) = e^{x-2t} \tag{19}$$

Example 2 (Adomian Decomposition Method)

Solve the following homogeneous telegraph equation

$$u_{xx} = u_{tt} + 4u_t + 4u \tag{20}$$

Subject to the conditions

$$\begin{aligned} u(0,t) = 1 + e^{-2t}, u_x(0,t) = 2 \\ u(x,0) = 1 + e^{2x}, u_t(x,0) = -2 \end{aligned} \tag{21}$$

Solution

Operate with L_x^{-1} on (20) and using the boundary condition gives

$$u(x,t) = 1 + e^{-2t} + 2x + L_x^{-1}(u_{tt} + 4u_t + 4u) \tag{22}$$

This gives;

$$\sum_{n=0}^{\infty} u_n(x,t) = 1 + e^{-2t} + 2x + L_x^{-1}((\sum_{n=0}^{\infty} u_n)_{tt} + 4(\sum_{n=0}^{\infty} u_n)_t + 4\sum_{n=0}^{\infty} u_n)` \tag{23}$$

The recursive relation is given by

$$u_0(x,t) = 1 + e^{-2t} + 2x$$

$$\vdots$$

$$u_{k+1}(x,t) = L_x^{-1}(u_{ktt} + 4u_{kt} + 4u_k), \ K \geq 0 \tag{24}$$

In view of (24) we obtain

$$\begin{aligned} u_0(x,t) &= 1 + e^{-2t} + 2x \\ u_1(x,t) &= L_x^{-1}(u_{0tt} + 4u_{0t} + 4u_0) \\ &= 2x^2 + \frac{4}{3}x^3 \\ u_2(x,t) &= L_x^{-1}(u_{1tt} + 4u_{1t} + 4u_1) \\ &\frac{2}{3}x^4 + \frac{4}{15}x^5 \end{aligned} \tag{25}$$

Other components can be computed in a similar manner. Consequently, the solution in a series form is given by

$$u(x,t) = e^{-2t} + \left(1 + 2x + \frac{1}{2!}(2x)^2 + \frac{1}{3!}(2x)^3 + \cdots\right) \tag{26}$$

So the exact solution

$$u(x,t) = e^{2x} + e^{-2t} \tag{27}$$

Example 1 (Using Laplace Adomian Decomposition Method)

Solve the following homogeneous telegraph equation

$$u_{xx} = u_{tt} + u_t - u$$

ISSN 1584 – 2665 (printed version); ISSN 2601 – 2332 (online); ISSN-L 1584 – 2665

Subject to the conditions

$$u(0,t) = e^{-2t}, u_x(0,t) = e^{-2x}$$

$u(x,0) = e^x, u_t(x,0) = -2e^x$

Solution

Applying Laplace Transform as follows;

$$L\{u_{xx}\} = L\{u_{tt}\} + L\{u_t\} - L\{u\}$$
$$s^2L\{u\} = L\{u_{tt}\} + L\{u_t\} - L\{u\} \quad (28)$$

Substituting the boundary conditions (14) into (28), we have

$$s^2L\{u\} - su(0,t) - u_x(0,t)$$
$$= L\{u_{tt}\} + L\{u_t\} - L\{u\}$$
$$s^2L\{u\} - s\{e^{-2t}\} - \{e^{-2t}\} \quad (29)$$
$$= L\{u_{tt}\} + L\{u_t\} - L\{u\}$$

Divide equation (29) by s^2

$$L\{u\} = \frac{1}{s}\{e^{-2t}\} - \frac{1}{s^2}\{e^{-2t}\}$$
$$= \frac{1}{s^2}L\{u_{tt}\} + \frac{1}{s^2}L\{u_t\} - \frac{1}{s^2}L\{u\} \quad (30)$$

Rearranging equation (30), we have;

$$L\{u\} = \frac{1}{s}\{e^{-2t}\} + \frac{1}{s^2}\{e^{-2t}\} + \frac{1}{s^2}L\{u_{tt}\} + \frac{1}{s^2}L\{u_t\} - \frac{1}{s^2}L\{u\} \quad (31)$$

Now, using the decomposition representation for the dependent variables;

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) \quad (32)$$
$$L\{u_0\} = \frac{1}{s}\{e^{-2t}\} + \frac{1}{s^2}\{e^{-2t}\} \quad (33)$$

Take the Inverse Laplace Transform of both sides of equation (33)

$$u_0 = e^{-2t} + xe^{-2t} \quad (34)$$

Hence,

$$L\{u_1\} = \frac{1}{s^2}L\{u_{0tt}\} + \frac{1}{s^2}L\{u_{0t}\} - \frac{1}{s^2}L\{u_0\} \quad (35)$$

Taking the inverse Laplace Transform on both sides of equation (35)

$$u_1 = \frac{1}{2!}x^2e^{-2t} + \frac{1}{3!}x^3e^{-2t} \quad (36)$$
$$L\{u_2\} = \frac{1}{s^2}L\{u_{1tt}\} + \frac{1}{s^2}L\{u_{1t}\} - \frac{1}{s^2}L\{u_1\} \quad (37)$$

Taking the inverse Laplace Transform on both sides of equation (37)

$$u_2 = \frac{1}{4!}x^4e^{-2t} + \frac{1}{5!}x^5e^{-2t} \quad (38)$$
$$L\{u_3\} = \frac{1}{s^2}L\{u_{2tt}\} + \frac{1}{s^2}L\{u_{2t}\} - \frac{1}{s^2}L\{u_2\} \quad (39)$$
$$u_3 = \frac{1}{6!}x^6e^{-2t} + \frac{1}{7!}x^7e^{-2t} \quad (40)$$

And so on;

In general;

$$L\{u_{k+1}\} = \frac{1}{s^2}L\{u_{ktt}\} + \frac{1}{s^2}L\{u_{kt}\} - \frac{1}{s^2}L\{u_k\}$$

Combining the results obtained in equations (34), (36), (38) and (40), we get;

$$u(x,t) = u_0 + u_1 + u_2 + u_3 + \cdots$$
$$u(x,t) = e^{-2t}\left(1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \cdots\right)$$

Which gives the exact solution in the form $u(x,t) = e^{x-2t}$ as obtained in equation (19)

Example 2 (Laplace Adomian Decomposition Method)

Solve the following homogeneous telegraph equation

$$u_{xx} = u_{tt} + 4u_t + 4u$$

Subject to the conditions

$$u(0,t) = 1 + e^{-2t}, u_x(0,t) = 2$$

$u(x,0) = 1 + e^{2x}, u_t(x,0) = -2$

Solution

Applying Laplace Transform as follows

$$L\{u_{xx}\} = L\{u_{tt}\} + L\{4u_t\} + L\{4u\}$$
$$s^2L\{u\} = L\{u_{tt}\} + L\{4u_t\} + L\{4u\} \quad (41)$$

ISSN 1584 – 2665 (printed version); ISSN 2601 – 2332 (online); ISSN-L 1584 – 2665

Substituting the boundary condition (21) into (41), we have

$$s^2L\{u\} - su(0,t) - u_x(0,t) = L\{u_{tt}\} + L\{4u_t\} + L\{4u\}$$

$$s^2L\{u\} - s\{1 + e^{-2t}\} - 2 = L\{u_{tt}\} + L\{4u_t\} + L\{4u\} \tag{42}$$

Divide equation (42) by s^2

$$L\{u\} - \frac{1}{s}L\{1 + e^{-2t}\} + \frac{1}{s^2}L\{2\} = \frac{1}{s^2}L\{u_{tt}\} + \frac{1}{s^2}L\{4u_t\} + \frac{1}{s^2}\{4u\} \tag{43}$$

Rearranging equation (43), we have

$$L\{u\} = \frac{1}{s}\{1 + e^{-2t}\} + \frac{1}{s^2}\{2\} + \frac{1}{s^2}L\{u_{tt}\} + \frac{1}{s^2}\{4u_t\} + \frac{1}{s^2}\{4u\} \tag{44}$$

Now, using the decomposition representation for dependent variable u;

$$u(x,t) = \sum_{n=0}^{\infty} u_n\,(x,t) \tag{45}$$

$$L\{u_0\} = \frac{1}{s}\{1 + e^{-2t}\} + \frac{1}{s^2}\{2\} \tag{46}$$

Take the Inverse Laplace Transform of both sides of equation

$$u_0 = 1 + e^{-2t} + 2x \tag{47}$$

Hence,

$$L\{u_1\} = \frac{1}{s^2}L\{u_{0tt}\} + \frac{1}{s^2}L\{4u_{0t}\} + \frac{1}{s^2}L\{4u_0\} \tag{48}$$

Tking the inverse laplace transform on both sides of the equation

$$u_1 = 2x^2 + \frac{4}{3}x^3 \tag{49}$$

$$\{Lu_2\} = \frac{1}{s^2}L\{u_{1tt}\} + \frac{1}{s^2}L\{4u_{1t}\} + \frac{1}{s^2}L\{4u_1\} \tag{50}$$

$$u_2 = \frac{2}{3}x^4 + \frac{4}{15}x^5 \tag{51}$$

And so on

In general;

$$L\{u_{k+1}\} = \frac{1}{s^2}L\{u_{ktt}\} + \frac{1}{s^2}L\{4u_{kt}\} - \frac{1}{s^2}L\{4u_k\}$$

Combining the results obtained in equations (47), (49) and (51), we get

$$u(x,t) = u_0 + u_1 + u_2 + \cdots$$

$$u(x,t) = e^{-2t}\left(1 + 2x + \frac{1}{2!}(2x)^2 + \frac{1}{3!}(2x)^3 + \cdots\right) \tag{52}$$

Which gives the exact solution in the form $u(x,t) = e^{2x} + e^{-2t}$ as obtained in equation (27).

4. RESULTS AND DISCUSSION

The telegraph equation is the focus of this work. The work focuses on two analytical methods for solving the telegraph equation. It was observed that the two analytical methods are valid and consistent as the solution obtained from the two analytical methods (Adomian Decomposition Method (ADM) and Laplace Adomian Decomposition Method (LADM)) provided the same result. The methods are accurate and requires less economization of terms. It therefore shows that the two methods converges rapidly and can be used interchangeably. The two methods can be used to solve other nonlinear partial differential equations as the results obtained using telegraph equations can be applied in such cases.

5. CONCLUSION

The telegraph equation is one of the many nonlinear partial differential equations. Like any other nonlinear partial differential equations, it involves variables and their partial derivatives. However, in this study, we try to compare two analytical methods to solve the telegraph equation. The two methods were valid and consistent and can be applied to solve other nonlinear partial differential equations. Two examples were analyzed and we used Adomian Decomposition Method (ADM) for the first two and then used Laplace Adomian Decomposition Method (LADM) for the same two examples. The results obtained demonstrate a great relationship as the results are the same. This two analytical methods can be used to solve other nonlinear partial differential equation because of the consistencies in the results obtained above. It is suggested that the two methods can be used in solving more practical problems ranging from quantum physics, fluid mechanics and even environmental studies.

References

[1] Thompson (2013) Jeremy Gay's Henri Poincare: A Scientific Biography, Princeton University Press
[2] Malfliet M (1992) Solitary wave equations. American Journal of Physics 60. 650–654
[3] Nassar H.A., Abdel–Razek M.A., Seddeek, A.K (2011) Expanding the tanh–function method for solving nonlinear equations. Applied Mathematics 2. 1096–1104

ISSN 1584 – 2665 (printed version); ISSN 2601 – 2332 (online); ISSN-L 1584 – 2665

[4] Mohiud–Din S.T, Noor M.A, Noor K.I, Hosseini M.M (2010) solutions of singular equations by He's variational iteration method. International Journal of Nonlinear Sciences and Numerical Simulation II 81–86
[5] Mohiud–Din S.T (2007) Homotopy perturbation method for solving fourth–order boundary value problems. Mathematical Problems in Engineering 2: 1–15
[6] Ma W.X, You Y (2004) Rational Solution of the Toda lattice equation in Casoration form. Chaos, Solitons and Fractals 22; 395–406
[7] Ma W.X, Wu H.Y, He J.S (2007) Partial Differential Equations possessing Frobenius integrable decompositions. Physics letters A. 364; 29–32
[8] Islam M.E, Khan K, Akbar M.A, Islam R (2013) Travelling wave solution of nonlinear evolution equation via enhanced (G'/G) – expansion method. Bangladesh Mathematical Society 33:83–92
[9] Islam SMR, Khan K, Akbar M.A Islam R (2013) Enhanced (G'/G) – Expansion method to find the exact complexion Solutions of (3+1) – Dimensional Zakhrov – Kuznetsov Equation. Global Journals Incorporated 13(8)
[10] Muhammad Minarul Islam and Mohammad Sanjeed Hassan (2018) A study on exact solution of the telegraph equation by (G'/G) – expansion method. African Journal of Mathematics and Computer Science Research vol 11(7); 103–108
[11] H. Jafari, C.M. Khalique, M.Khan and M. Ghasemi (2011); A two–step laplace decomposition method for solving nonlinear partial differential equations. International Journal of the Physical Sciences Vol. 6(16); 4102–4109
[12] E.K. Jaradat, A.D Aloqali, W. Alhabashneh (2018) using Laplace Decomposition Methods to solve Nonlinear Klein–Gordon equation. U.P.B. Sci. Bull. Series D, Vol. 80, Iss .2.
[13] Mohammed E.A. Rabie (2015) Solvability of nonlinear Klein–Gordon equation by Laplace Decomposition Method. African Journal of Mathematics and Computer Science Research vol 8(4) : 37–42
[14] Farah F. Ghazi and Tawfia L.N.M. (2020) coupled Laplace–Decomposition Method for solving Klein–Gordon Equation. International Journal of modern Mathematical Sciences 18(1); 31–41

ISSN 1584 – 2665 (printed version); ISSN 2601 – 2332 (online); ISSN–L 1584 – 2665