

ON THE TRANSFER AND STIFFNESS MATRICES IN GENERALIZED THERMOELASTICITY IN LAYERED MEDIA

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Abstract: A theoretical exploration is presented for the obtaining a transfer matrix and using this technique with stiffness matrix for obtaining a solution for dispersion curves in heat conducting n -layered isotropic in the context of generalized theory of thermoelasticity. This paper discusses transfer matrix technique and obtained stiffness matrix for heat conducting isotropic layered in generalized thermoelasticity with thermal relaxation. Some special cases have also been deduced and discussed from the obtained result.

Keywords: transfer matrix, stiffness matrix, thermal relaxation, generalized thermoelasticity

1. INTRODUCTION

Composite materials are employed in numerous structural components as such materials are perfect for structural applications, comprising motorized parts, infrastructures, science, aerospace and engineering structures. Laminated plate structures are also the most significant applied structure. Composite materials are employed in numerous structural components, comprising motorized parts, infrastructures, science, aerospace and engineering structures.

Studies of the elastic waves in multilayered and laminated media are well documented in many references [1–2]. Transfer matrix technique [3–5] is a reliable technique for wave propagation analysis in layered media. Liu et al. [6] studied the wave propagation in arbitrary anisotropic laminates on the basis of an exact theory. Nayfeh [7] covered the detailed investigation of wave propagation in layered anisotropic media and covered transfer matrix techniques. Transfer and stiffness matrices are well documented in [8–12] where a mathematical formulation of transfer matrix technique is discussed and the instability of transfer matrix techniques by using stiffness matrix techniques elimination is also studied.

Lord and Shulman [13] extended the coupled theory of thermoelasticity by introducing the thermal relaxation time in the constitutive equations. This theory, which eliminates the paradox of infinite velocity of heat propagation, is called generalized theory of thermoelasticity. Thermoelasticity with second sound and hyperbolic thermoelasticity is well documented and reviewed by Chandrasekharaiah [14–15]. Propagation of waves in layered anisotropic media and laminated composites in generalized thermoelasticity is studied in detail by Verma [16, 17].

The modified theory of thermoelasticity report a finite nature of disturbance propagation is familiarized by introducing thermal relaxation time constants into the heat conduction equation. This theory of thermoelasticity is named generalized theory of thermoelasticity, which also take into account of the coupling of temperature and strain fields.

In this paper in generalized theory of thermoelasticity a theoretical exploration is presented for the obtaining a transfer matrix and using this technique with stiffness matrix for obtaining a solution for dispersion curves in heat conducting n -layered isotropic media. Stiffness matrix is also obtained and discussed from transfer matrix technique for heat conducting isotropic layered with thermal relaxation.

2. FORMULATION

Consider a heat conducting layered plate consisting of homogeneous n -layers rigidly bonded at their interfaces. The problem is to obtain a transfer matrix and then combining it with stiffness matrix for obtaining a solution for dispersion curves. Using two-dimensional coordinate systems (x, z) which have its origin at the bottom layer of the plate such that x denotes the propagation direction and z is the normal to the interfaces. Hence layered plane will then occupy the space $0 \leq z \leq h$ where h denotes the total thickness of the plate. Since the plate is made of n layers, k th layer will then have its local coordinates x_k and z_k with local origin at bottom surface. Hence each layer occupy the space $0 \leq z_k \leq h_k$ where h_k is its thickness. With this choice of co-ordinate system the equation of motion and heat conduction for each layer are:

$$\begin{aligned} \mu[u_{,xx} + u_{,zz}] + (\lambda + \mu)(u_{,xx} + w_{,zz}) &= \rho\ddot{u} + \gamma T_{,x}, \\ \mu(w_{,xx} + w_{,zz}) + (\lambda + \mu)(u_{,xx} + w_{,zz}) &= \rho\ddot{w} + \gamma T_{,z}, \end{aligned} \quad (1)$$

$$K[T_{,xx} + T_{,zz}] + \rho C_e (\dot{T} + \tau_0 \ddot{T}) = \gamma T_0 [(\dot{u}_{,x} + \dot{w}_{,z} + \tau_0 (\ddot{u}_{,x} + \ddot{w}_{,z}))].$$

The comma notation is used for spatial derivatives and the superposed dot denotes time differentiation. Symbols λ and μ are Lamé's constants, ρ is the density, τ_0 is the thermal relaxation time, α_i the coefficient of thermal expansion $\gamma = (3\lambda + 2\mu)\alpha_i$ is the thermoelastic coupling constant and all other symbols have their usual meanings as in [3].

3. ANALYSIS

For waves whose projected wave vector is along the x-axis, equations (1) admit the formal solutions

$$(u, w, T) = (U_1, U_2, U_3) \exp[i\xi(x + z\alpha - ct)] \quad (2)$$

where ξ is the wave number, U_1, U_2 and U_3 are the constant amplitudes related to displacements and temperature, c is the phase velocity ($= \omega/\xi$), ω is the circular frequency, α is the ratio of the z and x-directions wave numbers. This choice of solutions to equations (1) leads to the coupled equations

$$M_{mn}(\alpha)U_n = 0 \quad m, n = 1, 2, 3. \quad (3)$$

where the summation convention is implied, and

$$\begin{aligned} M_{11} &= c_2\alpha^2 + 1 - \zeta^2, \quad M_{13} = c_3\alpha, \quad M_{14} = 1, \quad M_{31} = M_{13}, \quad M_{33} = c_2 + \alpha^2 - \zeta^2, \quad M_{34} = \alpha, \\ M_{41} &= i\xi c \omega^* \zeta^2 \tau^* \varepsilon_1, \quad M_{43} = i\xi c \omega^* \zeta^2 \alpha \tau^* \varepsilon_1, \quad M_{44} = \omega^* \zeta^2 \tau - (1 + \alpha^2), \quad c_3 = 1 - c_2 \\ \varepsilon_1 &= \frac{T_0 \gamma^2}{\rho C_e (\lambda + 2\mu)}, \quad \omega^* = C_e (\lambda + 2\mu) / K, \quad \tau^* = (\tau_0 \delta_{1k} + i/\xi c)(\tau_1 + i/\xi c), \quad \tau = (\tau_0 + i/\xi c), \end{aligned} \quad (4)$$

$$c_2 = \mu / (\lambda + 2\mu), \quad \zeta^2 = \rho c^2 / (\lambda + 2\mu). \quad (5)$$

The existence of non-trivial solution for U_1, U_2 and U_3 determinant in equations (3), vanishes and yields the polynomial equation

$$(\zeta^2 - c_2\alpha^2 - c_2)(\alpha^4 + A\alpha^2 + B) = 0, \quad (6)$$

where $A = 2 - [(\omega^* \tau + 1) - i\xi c \tau^* \varepsilon_1] \zeta^2$, $B = [(\zeta^2 - 1)(\zeta^2 \omega^* \tau - 1) + i\xi c \zeta^2 \tau^* \varepsilon_1]$.

Solving (6) for the six roots of α and using superposition results in the following formal solution relating the displacements, temperature, thermal stresses and temperature gradient within a layer to its wave amplitudes.

$$\begin{bmatrix} u \\ w \\ T \\ \bar{\sigma}_{zz} \\ \bar{\sigma}_{xz} \\ \bar{T}' \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & -\frac{1}{\alpha_3} & \frac{1}{\alpha_3} \\ \alpha_1 & -\alpha_1 & \alpha_2 & -\alpha_2 & \alpha_3 & \alpha_3 \\ S_1 & S_1 & S_2 & S_2 & 0 & 0 \\ D_1 & D_1 & D_1 & D_1 & D_3 & D_3 \\ D_4 & -D_4 & D_5 & -D_5 & D_6 & -D_6 \\ D_7 & -D_7 & D_8 & -D_8 & 0 & 0 \end{bmatrix} \begin{bmatrix} A_1 E_1 \\ A_2 E_2 \\ A_3 E_3 \\ A_4 E_4 \\ A_5 E_5 \\ A_6 E_6 \end{bmatrix} E, \quad (7)$$

here α_1^2, α_2^2 are roots of $\alpha^4 + A\alpha^2 + B = 0$ and $\alpha_3^2 = \frac{c^2}{c_T^2} - 1$, $E_q = e^{i\xi\alpha_q z}$, $E = e^{i\xi(x-ct)}$, $q = 1, 2 \dots 6$.

α_1^2, α_2^2 Corresponds to longitudinal and thermal waves whereas α_3^2 corresponds to transverse wave which is not affected by the temperature variations. Also

$$D_1 = c_2 \left(\frac{c^2}{c_T^2} - 2 \right), \quad D_3 = -2c_2, \quad D_4 = 2c_2\alpha_1, \quad D_5 = 2c_2\alpha_2, \quad D_6 = \frac{c_2}{\alpha_3} \left(\frac{c^2}{c_T^2} - 2 \right), \quad D_7 = \alpha_1 S_1, \quad D_8 = \alpha_2 S_2, \quad c_T^2 = \frac{\mu}{\rho},$$

$$S_q = -\frac{\omega^* \tau^* \zeta^2 \varepsilon_1}{(1 + \alpha_q^2 - \omega^* \tau^* \zeta^2)} (1 + \alpha_q^2), \quad q = 1, 2, \quad \bar{\sigma}_{zz} = \frac{\sigma_{zz}}{i\xi}, \quad \bar{\sigma}_{xz} = \frac{\sigma_{xz}}{i\xi}, \quad \bar{T}' = \frac{T'}{i\xi}.$$

The continuity of displacement, temperature $D(z) = \{\bar{u}, \bar{w}, \bar{T}\}$ and stresses and the temperature gradient components $S(z) = \{\bar{\sigma}_{xz}, \bar{\sigma}_{zz}, \frac{\partial \bar{T}}{\partial z}\}$ at the interface

$$D^{(n)}(h_n) = D^{(n-1)}(h_{n-1}); \quad S^{(n)}(h_n) = S^{(n-1)}(h_{n-1}) \quad (8)$$

4. TRANSFER MATRIX

Specializing (7) to the upper and bottom surface of each layer, we can relate, after lengthy algebraic reductions and manipulations, the displacements, temperature, stresses and temperature gradient of the upper layer to those of the bottom as

$$P_j^{(u)} = [a_{ij}]_{6 \times 6} P_j^{(b)}, \quad (9)$$

where P_j is a column vector consisting of $u, w, T, \bar{\sigma}_{zz}, \bar{\sigma}_{xz}, \bar{T}$, and the superscripts (u) and (b) designate quantities defined at the upper and bottom of the jth layer respectively. By repeating the above process to each layer and invoking the continuity conditions on the upper and bottom of each layer to those of its neighbors we can finally relate the displacements, temperature, stresses and temperature gradient at the top (top of layer (n)) of the plate to those at the bottom of the plate (bottom of layer (1)) via the transfer matrix multiplications

$$A = A_n A_{n-1} \dots A_1 \quad (10)$$

which leads to global transfer matrix

$$P^{(u)} = AP^{(b)}, \quad (11)$$

$$P^{(u)} = \begin{bmatrix} u^{(n)} \\ w^{(n)} \\ T^{(n)} \\ \bar{\sigma}_{zz}^{(n)} \\ \bar{\sigma}_{xz}^{(n)} \\ \bar{T}^{(n)} \end{bmatrix}_{z=h}, P^{(b)} = \begin{bmatrix} u^{(1)} \\ w^{(1)} \\ T^{(1)} \\ \bar{\sigma}_{zz}^{(1)} \\ \bar{\sigma}_{xz}^{(1)} \\ \bar{T}^{(1)} \end{bmatrix}_{z=0} \quad A = [A_{ij}]_{6 \times 6} \quad (12)$$

where $P^{(u)}, P^{(b)}$ are the displacement, temperature, stresses and temperature gradient column vectors at the top $z = h$ and bottom $z = 0$ and $A = [A_{ij}]_{6 \times 6}$ of the total plate respectively.

Using the matrix equation (11) to rewrite the transfer matrix which is relating the layer properties and boundary conditions at the top and bottom surfaces with other layers.

$$\begin{bmatrix} D^{(u)} \\ S^{(u)} \end{bmatrix} = \begin{bmatrix} [A_{DD}] & [A_{DS}] \\ [A_{DS}] & [A_{SS}] \end{bmatrix} \begin{bmatrix} D^{(b)} \\ S^{(b)} \end{bmatrix}, P^{(b)} = \begin{bmatrix} D^{(b)} \\ S^{(b)} \end{bmatrix}, P^{(u)} = \begin{bmatrix} D^{(u)} \\ S^{(u)} \end{bmatrix} \quad (13)$$

In equation (13) matrix $A_{6 \times 6}$ is written in terms of sub matrices A_{SS}, A_{SD}, A_{SD} and A_{DD} are 3×3 matrices. For the kth layer and $A_k = X_k H_k X_k^{-1}$ is having diagonal matrix H_k with diagonal entices $E_l = \exp(i\xi\alpha_l x_3)$

whose determinant is equal to the product of its entries $\prod_{j=1}^6 E_j$.

where $P^{(u)}, z = \text{top}$ and $P^{(b)}, z = \text{bottom}$ are the displacement, temperature, stress and temperature gradient column vectors at the top and bottom of the total plate respectively.

5. STIFFNESS AND COMPLIANCE MATRIX

Equations (7) can be expressed in the matrix form as

$$\begin{bmatrix} S \\ D \end{bmatrix}^{(n)} = \begin{bmatrix} K_1 & K_2 \\ K_3 & K_4 \end{bmatrix}^{(n)} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}^{(n)} \quad (14)$$

Specialize (14) and write only the displacement and temperature matrix on the upper and lower surfaces of the layer n can be represented in the matrix form

$$\begin{bmatrix} S_{z=h_n} \\ S_{z=h_{n-1}} \end{bmatrix}^{(n)} = \begin{bmatrix} Q_1 & Q_2 \\ G_1 & G_2 \end{bmatrix}^{(n)} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}^{(n)} = N_S^{(n)} U^{(n)}, U^{(n)} = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}^{(n)} \quad (15)$$

Similarly specialize (14) and write only the stresses and temperature gradient matrix on the upper and lower surfaces of the nth layer can be represented in the matrix form

$$\begin{bmatrix} D_{z=h_n} \\ D_{z=h_{n-1}} \end{bmatrix}^{(n)} = \begin{bmatrix} Q_3 & Q_4 \\ G_3 & G_4 \end{bmatrix}^{(n)} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}^{(n)} = M_D^{(n)} U^{(n)} \quad (16)$$

Equations (15 and (16) relate the stresses and displacements on the nth layer surfaces to the wave displacement amplitudes $U^{(n)}$.

Now substituting the amplitude vector $U^{(n)}$ from Eq. (16) into (15) we obtain

$$\begin{bmatrix} S_{z=h_n} \\ S_{z=h_{n-1}} \end{bmatrix}^{(n)} = N_S^{(n)} (M_D^{(n)})^{-1} \begin{bmatrix} D_{h_n} \\ D_{h_{n-1}} \end{bmatrix}^{(n)} \quad (17)$$

Similarly Now substituting the amplitude vector $U^{(n)}$ from Eq. (15) into (16) we obtain

$$\begin{bmatrix} D_{z=h_n} \\ D_{z=h_{n-1}} \end{bmatrix}^{(n)} = M_D^{(n)} (N_S^{(n)})^{-1} \begin{bmatrix} S_{h_n} \\ S_{h_{n-1}} \end{bmatrix}^{(n)} \quad (18)$$

Matrix in (17)

$$S_T^{(n)} = N_S^{(n)} (M_D^{(n)})^{-1} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}^{(n)} \quad (19)$$

is defined as a layer stiffness matrix.

Matrix in (18)

$$D_C^{(n)} = M_D^{(n)} (N_S^{(n)})^{-1} = \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix}^{(n)} \quad (20)$$

is defined as a layer compliance matrix.

For stiffness matrix for bottom layer with the coordinate origin at the plate surface, only three waves $U_1^{(n)}$ propagate in the z^- direction from the surface to infinity. The displacements and stresses can be obtained from Eqs. (15) and (16) respectively: $D_{h_n}^{(1)} = Q_3^{(1)} A_1^{(1)}$ and $S_{h_n}^{(1)} = Q_1^{(1)} A_1^{(1)}$.

Therefore the stiffness matrix $M^{(1)}$ for a bottom layer is given by

$$S_T^{(1)} = Q_3^{(1)} (Q_1^{(1)})^{-1} A_1^{(1)} = M^{(1)} D_{h_n}^{(1)} \quad (21)$$

where $M^{(1)}$ is the stiffness matrix.

To obtain transfer matrix reorganize equations (14), representing the stresses, temperature gradient, displacements and temperature on the top and bottom surfaces of the nth layer as

$$\begin{bmatrix} S \\ D \end{bmatrix}_{z=0}^{(n)} = G_0^{(n)} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}^{(n)}, G_0^{(n)} = \begin{bmatrix} G_1 & G_2 \\ G_3 & G_4 \end{bmatrix}^{(n)} \quad (22)$$

$$\begin{bmatrix} S \\ D \end{bmatrix}_{z=h_n}^{(n)} = Q_{h_n}^{(n)} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}^{(n)}, Q_{h_n}^{(n)} = \begin{bmatrix} Q_1 & Q_2 \\ Q_3 & Q_4 \end{bmatrix}^{(n)} \quad (23)$$

From equations (22) and (23) eliminate $\begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$ we have

$$\begin{bmatrix} S \\ D \end{bmatrix}_{z=0}^{(n)} = (Q_{h_n}^{(n)})^{-1} G_0^{(n)} \begin{bmatrix} S \\ D \end{bmatrix}_{z=h_n}^{(n)} = Z^{(n)} \begin{bmatrix} S \\ D \end{bmatrix}_{z=h_n}^{(n)}$$

Following [9] the relation between transfer matrix $Z^{(n)}$ and stiffness matrix $M^{(n)}$ is given by

$$P^{(n)} = \begin{bmatrix} -(S_{12})^{-1} S_{11} & (S_{12})^{-1} \\ S_{21} - S_{22} (S_{12})^{-1} S_{11} & S_{22} (S_{12})^{-1} \end{bmatrix}^{(n)} \quad (24)$$

Thus the layer transfer $P^{(n)}$ and stiffness $M^{(n)}$ matrices are formed from the same.

The stiffness matrix can also be represented through the transfer matrix elements

$$S^{(n)} = \begin{bmatrix} -(P_{12})^{-1} P_{11} & (P_{12})^{-1} \\ P_{21} - P_{22} (P_{12})^{-1} P_{11} & P_{22} (P_{12})^{-1} \end{bmatrix}^{(n)} \quad (25)$$

6. COUPLED THERMOELASTICITY

This case corresponds to no thermal relaxation time, i.e. $\tau_0 = 0$. Following the above procedure as in the above case, we again arrived at equations similar to (24) and (25).

The global transfer matrix is calculated by multiplying the transfer matrix of individual layers consecutively. And to satisfy thermally insulated and stress-free layered plate condition for the whole laminate equation

(13) are set to zero i.e. $S(z) = \{\bar{\sigma}_{xz}, \bar{\sigma}_{zz}, \frac{\partial \bar{T}}{\partial z}\} = 0$, we obtain the characteristic equation as

$$\det[A_{DS}] = 0 \quad (26)$$

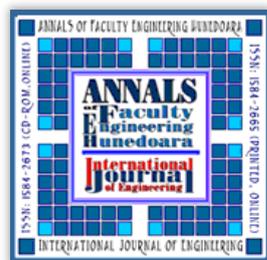
Equation (26) is solved to find dispersion phase velocities versus wavenumbers or frequencies. Results obtained are in agreement with the corresponding results obtained by [18,19].

7. CONCLUSION

The paper fleetingly aimed at the mathematical formulation of transfer and stiffness matrices for a heat conducting multilayered media in generalized thermoelasticity with one relaxation time. Transfer matrix technique is one of the consistent procedures for wave propagation analysis in layered media as its advantage is that it condenses the multi-layered system into less numbers of equations, whereas SH waves decoupled and remaining equations relating the boundary conditions at the bottom and the last interfaces. Transfer matrix technique removes all other in-between interfaces, which reduces a lot of computational time and complexity. Hence, this technique is advantageous with a drawback that it may agonizes is its algebraic uncertainty of the result at large values of frequency and thickness products. Other various advantages of this technique are to solve characteristic equations for the free waves on single layered plate and on the periodic media. Transfer matrix technique can be used in solving various continuum mechanics forthcoming work.

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