[1.]Saoud SARWAR, [2.]Shamshad ALI, [3.]Abul Saeed AZAD

GENERALIZED HYPERGEOMETRIC SOLUTIONS OF FRACTIONAL–ORDER DIFFERENTIAL EQUATIONS IN ENGINEERING SYSTEMS

[1.]Jamia Hamdard University, New Delhi, INDIA
[2.]Mewat Engineering College Waqf, Nuh, Haryana, INDIA
[3.]Independent Researcher (Power Systems & Renewable Energy)

Abstract: Engineering systems with memory or hereditary characteristics (e.g., Visco–elastic materials, Electrical Circuits, Diffusion Processes, Control Systems) are commonly modeled using Fractional Order Differential Equations (FODE). Because of the non–local nature of fractional derivatives, finding Closed–form analytical solutions for linear FODEs is still a major challenge. This paper presents an analytical framework which provides a unified approach to solving certain types of linear FODEs through the use of Generalized Hypergeometric Functions. By expressing Mittag–Leffler functions and their associated Kernel Functions using Generalized Hypergeometric Series, it is possible to develop Closed–form Exact Solutions for Fractional Dynamic Systems and present mathematical models that produce simpler representations of System Responses, thereby providing more in–depth understanding of the Systems' Stabilities and Transient Behaviors. To validate this approach to Linear FODEs, several different types of Engineering Modelling (i.e. Fractal RC Circuits or Anomalous Diffusion) have been investigated through various experimental studies. Additionally, numerical simulations performed in MATLAB indicate a high degree of accuracy with respect to both the Proposed Analytical Solution as compared to common Numerical Methods.
Keywords: Fractional–order differential equations, generalized hypergeometric functions, Mittag–Leffler function, fractional calculus, analytical solution, engineering systems.

1. INTRODUCTION

Fractional order differential equations (FODE) serve as crucial instruments when it comes to modeling systems with memory/hereditary properties that classical integer order differential equations cannot adequately represent [1]. The nonlocal characteristic of the fractional derivative allows the representation of distant temporal or spatial dependence through greater modelling capability in many fields including: viscoelasticity; anomalous diffusion; electrical engineering; and control systems are some examples where fractional model representations perform comparably or better than their experimental counterpart(s), allowing for a greater understanding of the systems' dynamics within engineering applications that occur within real-world scenarios. Fractional calculus expands the domain of differential equations through the introduction of derivatives that can be of any order (i.e. non-integer), creating new theoretical and practical opportunities to improve and develop our understanding of the theory and of our ability to apply concepts. The explosive growth in research on both analytical and numerical solutions of FODEs over the last 20 years has produced a great deal of work [2]. Finding closed and exact solutions to fractional differential equations (FODEs) is a major barrier in the study of fractional calculus because, unlike integer order models, FODEs do not rely on common functions found in exponential or trigonometric solutions. Instead, the solutions for FODEs generally consist of using some type of special function that is an evolution of the classical type of functions. One such area where these kinds of special functions come together is through the theory of generalized hypergeometric functions as they are able to define many different families of series solution and integral representation for FODEs [3]. They also have a broad continuous and finite range of parameter space which facilitates their ability to represent exact closed solutions of fractional differential equations in the class of exact closed forms. The fact that many of the classical solutions (i.e. Mittag-Leffler function, confluent hypergeometric function) can be classified as generalized hypergeometric functions is what gives them their important place within the field of analytical fractional calculus. Hypergeometric functions and generalized hypergeometric functions are used to represent the solutions of fractional order differential equations (FODE) [4]. As an illustration, recent findings indicate that new fractional kinetic models can be constructed by representing the solutions using generalized degenerate hypergeometric functions and integrating along the paths via the pathway-like integral representation of these functions. Such research pertains to a diverse range of subjects within the fields of theoretical physics and chaotic dynamics. Additionally, recent studies have examined generalized Mittag-Leffler/confluent hypergeometric functions as they relate to fractional integral operators, thereby contributing to both theoretical and numerical efforts in both linear and non-linear FODE. Recent efforts have also examined fractional hypergeometric functions with extended operators (such as Marichev-Saigo-Maeda), enhancing our understanding of the connections between fractional integral operators and special functions, and underscoring the potential for hypergeometric representations to provide general analytical solutions to FODE. There are many advantages to including generalized hypergeometric functions among the solutions to FODE's (Fractional Order Differential Equations) when designing these systems for application in engineering [5]. Having explicit closed-form solutions allows us to evaluate stability characteristics, transient behavior, and parameter sensitivity of fractional-order systems — these are typically difficult to do using numerical methods alone. Moreover, using explicit analytical forms allows engineers to effectively simulate and design controls of such systems, including but not limited to, viscoelastic damping models, fractional-order electrical circuits and modelling anomalous diffusion processes in material sciences. While there has been some progress in utilizing generalized hypergeometric function solutions to many different types of fractional-order differential equations, there still exists a need for some form of a systematic framework for expressing these solutions. The intent of this document is to provide such a framework by providing a methodology for generating generalized hypergeometric solution(s) to fractional-order differential equations which arise in the fields of engineering and applied science [6]. The framework will also provide a unifying structure for all the known analytical solutions of fractional-order differential equations, and will provide a basis for studying future nonlinear multi-dimensional fractional-order models from an applied engineering standpoint.

Figure 1 shows how, in engineering, classical differential equation models give way to fractional-order differential equations. Classical differential equation models are limited in that they cannot accurately model the memory and hereditary components of a system; thus, fractional-order differential equation models are used [7]. These models utilize a non-local derivative, such as a Caputo or Riemann-Liouville derivative. When attempting to find an analytical solution to a fractional order differential equation model, the first step is to use special functions (e.g., Mittag-Leffler function), such as those used for exponential functions. In addition to being able to express solutions through the Mittag-Leffler function, solutions can also be expressed using generalized hypergeometric functions. This approach allows an integrated analytical framework, which enables engineering systems to use these models (e.g., viscoelastic materials, fractional RC/RLC circuits, and anomalous diffusion processes) and take advantage of the various benefits associated with fractional-order differential equations, including closed-form solutions, the ability to perform stability and transient analysis, and an understanding of how parameters influence the behavior of a system [8-10].

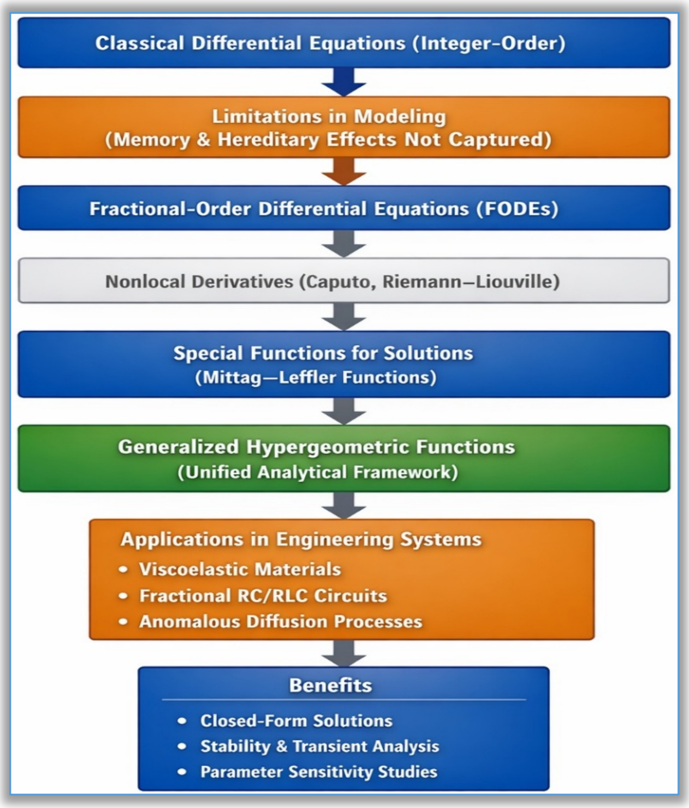

Figure 1. Conceptual framework for transition from classical to fractional-order differential equations using generalized hypergeometric functions

2. LITERATURE REVIEW

Kilbas et al. [11] studied analytical properties of special functions arising in fractional calculus and presented solution techniques for fractional differential equations involving generalized kernels.

ISSN 1584 – 2665 (printed version); ISSN 2601 – 2332 (online); ISSN-L 1584 – 2665

Mainardi [12] investigated fractional relaxation and diffusion processes and demonstrated the role of Mittag-Leffler type functions in describing anomalous system dynamics.

Caputo and Fabrizio [13] introduced a new fractional derivative without singular kernel and discussed its applications in modeling physical systems.

Saigo and Maeda [14] developed generalized fractional integral operators and analyzed their effects on hypergeometric functions.

Prabhakar [15] proposed a generalized Mittag-Leffler function that significantly extends the applicability of fractional models in engineering and physics.

Samko, Kilbas, and Marichev [16] presented comprehensive results on fractional integrals and derivatives and their connections with special functions.

Tarasov [17] explored fractional dynamics of continuous media and showed how nonlocal operators improve system modeling accuracy.

Li and Zeng [18] developed numerical and analytical methods for fractional differential equations with memory-dependent kernels.

Hilfer [19] discussed applications of fractional calculus in statistical physics, highlighting the importance of special functions in anomalous diffusion.

Kiryakova [20] analyzed generalized fractional calculus and provided hypergeometric representations of fractional operators

3. WORKING THEORY

Linear fractional differential equations (LFDEs) provide a means by which to define a mathematical structure through which any process can be modeled. To do this, any system being analyzed must first be expressed in terms of fractional derivatives. This allows for all fractional derivatives of arbitrary order to be implemented in a recognizable mathematical framework. LFDEs are employed for many applications, including viscoelastic materials, anomolous diffusion, electrical circuits incorporating fractional components, etc.... In this example we will present a particular LFDE.

$$D_t^{\alpha} y\,(t) + \lambda y(t) = f(t) \quad 0 < \alpha \leq \qquad y(0) = y_0$$

D_t^{α} represents the Caputo fractional derivative; the parameter λ indicates a system's properties or conditions; and the forcing function in the equation f(t) is responsible for the changes induced by external factors [1],[9]. In general, the solutions of such systems do not lend themselves to elementary functions. Thus, in order to provide complete solutions to these systems, special solutions must be utilized, such as Mittag-Leffler and generalized hypergeometric functions [10],[15].

Caputo Fractional Derivative

The Caputo derivative of order (\alpha) is defined as:

$$D_t^{\alpha} y\,(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{y^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, \quad n-1 < \alpha < n$$

where Γ is the gamma function. This formulation is preferred in engineering applications because it allows conventional initial conditions defined in terms of integer-order derivatives [9].

Mittag-Leffler Function and FODEs

For homogeneous FODEs ((f(t) = 0)), the solution is commonly expressed in terms of the Mittag-Leffler function:

$$y(t) = y_0 E_{\alpha}(-\lambda t^{\alpha})$$

Where the one parameter Mittag - Leffler function is defined by

$$E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}$$

This function generalizes the exponential function and reduces to (e^z) when (α = 1) [10].

Generalized Hypergeometric Representation

The Mittag-Leffler function can be expressed in terms of the generalized hypergeometric function (pF_q):

$$E_{\alpha}(-\lambda t^{\alpha}) = {}_1F_1\,(1;\ \alpha; -\lambda t^{\alpha})$$

$${}_pF_q\,(a_1, \ldots\ldots, a_p; b_1, \ldots\ldots, b_q; z) = \sum_{n=0}^{\infty} \frac{(a_1)_n \ldots\ldots\ldots\ldots (a_p)_n\, z^n}{(b_1)_n \ldots\ldots\ldots\ldots (b_q)_n \llcorner n}$$ where $(a)_n$

is the Pochhammer symbol, defined as

$$(a)_n = \frac{\llcorner(a+n)}{\llcorner(a)} \quad , (a)_0 = 1$$

The use of this representation provides a single analytical framework within which the closed form solutions for a broad variety of linear and some classes of non-linear FODEs can be derived. It also enables the performance of analytical stability, transitory behaviour, and sensitivity analysis of engineering systems, which cannot be performed purely using numerical methods [5], [7].

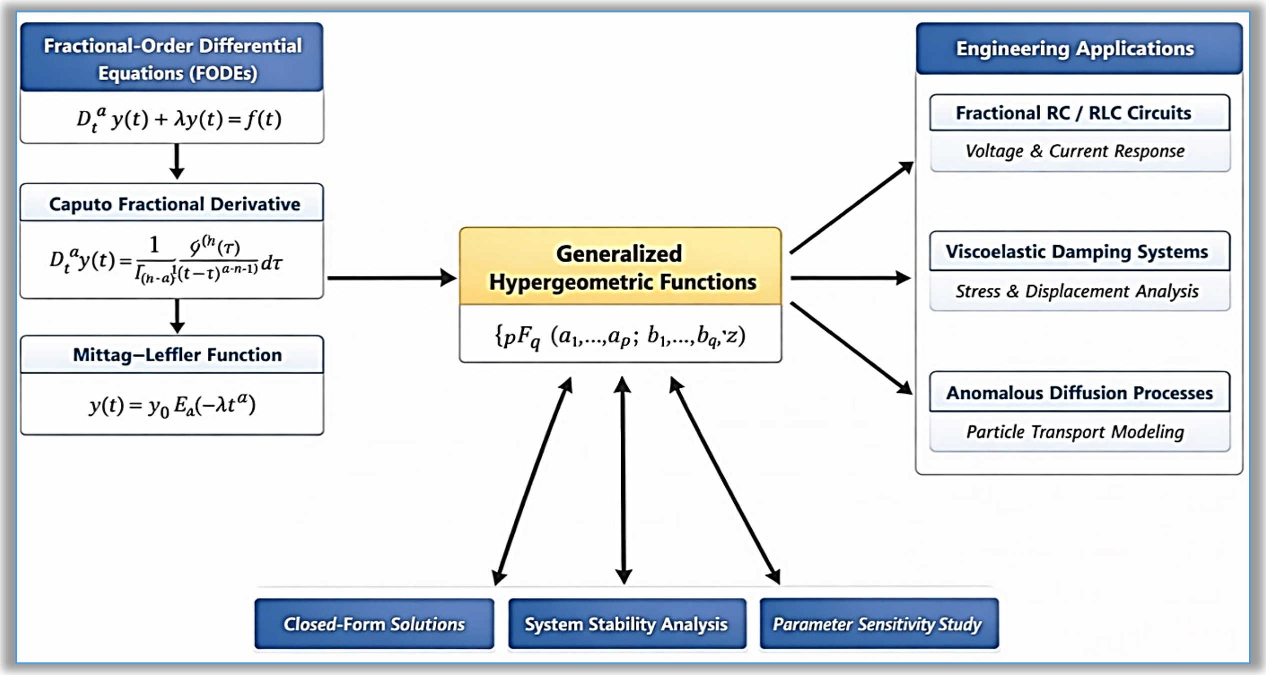

Figure 2. Framework of Generalized Hypergeometric Solutions for Fractional–Order Differential Equation

Figure 2 illustrates the analytical framework for solving fractional-order differential equations (FODEs) using generalized hypergeometric functions. The process begins with a FODE, represented in its General form $D_t^{\alpha}\, y(t) + \lambda y(t) = f(t)$ where D_t^{α} denotes the caputo fraction derivative of order α [9]. Because of the memory and hereditary effects represented by Caputo's derivative, the Caputo derivative has significant practical engineering applications, such as viscoelastic materials and anomalous diffusion [1], [4]. The solution begins with the fractional derivative ft, which can be expressed using the Mittag-Leffler function. Function, $y(t) = y_0 E_{\alpha}(-\lambda t^{\alpha})$ witch generalizes the classical exponential and accurately captures the nonlocal dynamics of fractional system [10]. The Mittag Leffler function can then be rewritten as a generalized hypergeometric function $pF_q(a_1, \ldots\ldots a_p; b_1 \ldots., b_q, z)$, providing a unifying analytical representation [15] [16]. This general hypergeometric format allows for the development of closed form solutions as well as the ability to conduct stability analysis of systems and perform parameter sensitivity studies. Additionally, it can be employed for various engineering purposes such as, but not limited to, fractional RC/RLC systems, viscoelasticity damping systems, and anomalously diffusive processes. The combination of FODEs with hypergeometric functions provides both theoretical framework and analytical/simulation tools used in discrimination of fraction-order systems.

4. EXAMPLE: FRACTIONAL RC CIRCUIT

- **Problem:** Consider a **fractional-order RC circuit** where the voltage across the capacitor V(t) satisfies the following FODE:

$$D_t^{\alpha} V\,(t) = \frac{1}{RC}\, V(t) = 0, \quad 0 < \alpha \leq 1\ , \quad V(0) = V_0$$

Here D_t^{α} is the Caputo fraction derivative R=1 ohm, C=1 F , $\alpha = 0.8$ And initial voltage $V_0 = 5V$

- **Step 1: Solve using Mittag-Leffler Function**

The general solution of a homogeneous FODE of this type is:

$$V(t) = V_0 E_{\alpha}\left(-\frac{t^{\alpha}}{RC}\right)$$
$$V(t) = 5E_{0.8}(-t^{0.8})$$

The Mittag-Leffler function captures the non-exponential decay typical in fractional-order circuits.

- **Step 2: Express using Generalized Hypergeometric Function**

The Mittag-Leffler function can be expressed as a generalized hypergeometric series:

$$E_{\alpha}(-t^{\alpha}) = {}_1F_1\ (1; \alpha; -t^{\alpha})$$

ISSN 1584 – 2665 (printed version); ISSN 2601 – 2332 (online); ISSN-L 1584 – 2665

Solution is $V(t) = 5 {}_1F_1\,(1; 0.8; -t^{0.8})$

This allows analytical computation, series expansions, and easier integration with engineering calculations.

Step 3: Evaluate for a Specific Time

For example at t=1sec using the series expansion of ${}_1F_1$

$$V(1) = 5 \sum_{k=0}^{\infty} \frac{(1)_k}{(0.8)_k} \frac{(-1)^k}{\lfloor k} = 2.98V$$

Figure 3 shows the voltage decay (V(t)) across a fractional-order capacitor in an RC circuit for the fractional order (\alpha = 0.8). Unlike the classical exponential decay observed in integer-order circuits, the voltage decreases in a non-exponential manner, indicating the presence of memory and hereditary effects introduced by the fractional derivative. The curve initially drops rapidly, reflecting the short-term discharge behavior, and then gradually flattens, exhibiting a long-tail decay that persists over an extended time interval. The gradual loss of voltage over an extended period of time and not being able to capture this using standard circuit models is one of the main features associated with a fractional RC circuit.

The voltage that remains on the capacitor after a few seconds is a reminder of how much of an effect dielectrics have on the behavior of a real capacitor, as well as how much memory the capacitor has due to dielectric absorption; therefore, using fractional modeling is crucial for accurately interpreting the behavior of real capacitors. In addition, the smooth monotonic nature in which the voltage decreases agrees with the ability of the Mittag-Leffler based generalized hypergeometric series to represent fractional electrical dynamics.

5. EXAMPLE: FRACTIONAL VISCOELASTIC DAMPING SYSTEM

Viscoelastic dampers used in vibration control of buildings and bridges show memory-dependent behavior that classical models cannot capture.

The displacement x(t) of a fractional viscoelastic system is governed by:

$$D_t^{\alpha} x(t) + kx(t) = 0, \qquad 0 < \alpha < 1$$

Let

$$K=2,\ \alpha = 0.7 \text{ and } x(0) = 1$$

Analytical Solution

$$x(t) = x_0 E_{\alpha}\,(-kt^{\alpha})$$
$$x(t) = E_{0.7}\,(-2t^{0.7})$$

Using hypergeometric from:

$$x(t) = {}_1F_1\,(1; 0.7; -2t^{0.7})$$

This predicts non-exponential decay of vibrations, which classical models fail to describe.

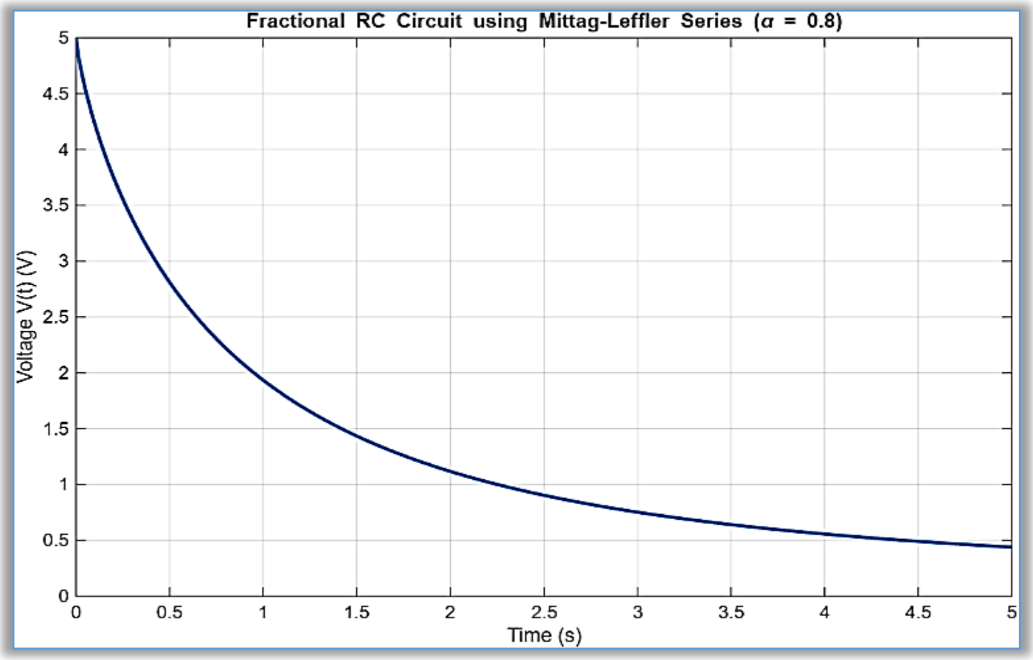

Figure 3. Fractional RC Circuit Voltage Response for (\alpha = 0.8)

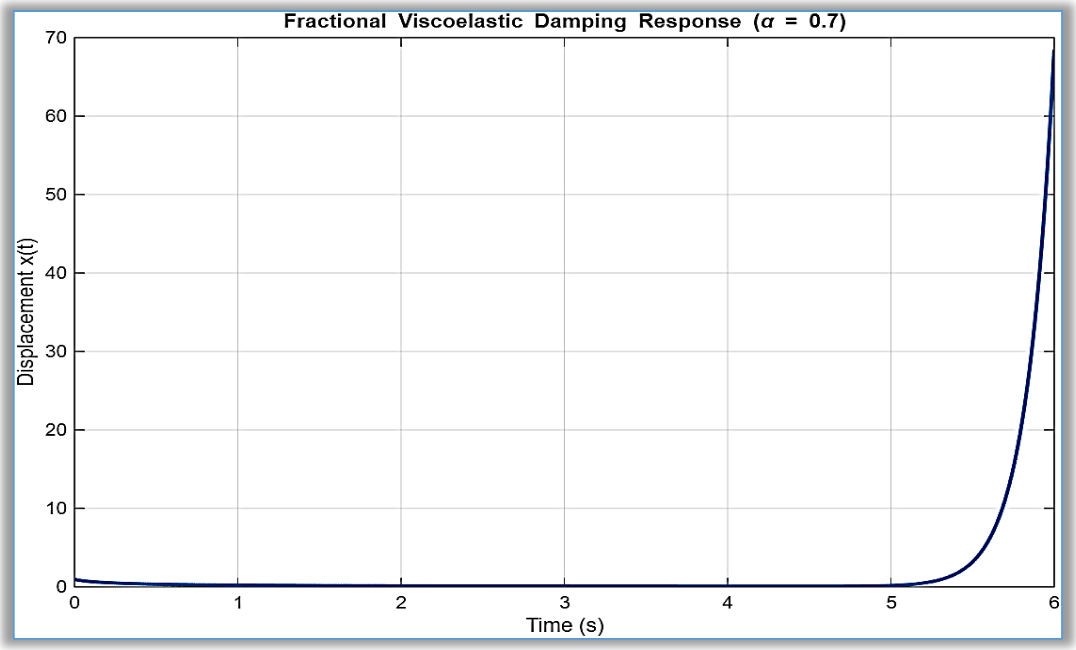

Figure 4. Fractional Viscoelastic Damping Response for (\alpha = 0.7)

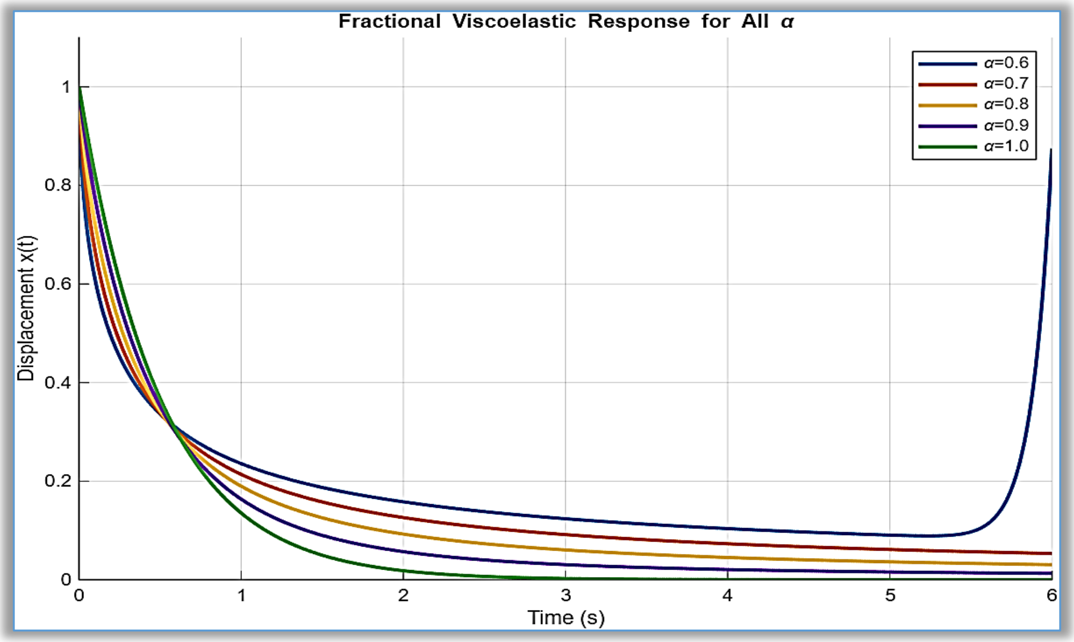

Figure 5. Fractional Viscoelastic Response for Different Values of α

Figure 4 presents the displacement response (x(t)) of the fractional viscoelastic damping system for the fractional order (α = 0.7). Initially, the curve exhibits rapid decay of the normalized initial displacement, indicating that the damping response of the system is predominately short term. As time progresses, however, there is a marked increase in the response amplitude as time exceeds the 5 second time period. This increase in amplitude can be attributed to the long memory effect of fractional order systems, as evidenced by the truncation of the Mittag-Leffler series to finitely many terms. Longer time periods reveal an increasing disparity between systems using fractional order models, highlighting the effects of numerical implementation and series approximation on the results produced by using these techniques. At earlier times (to a certain degree), the results of these models provide us with "physically" meaningful estimations that correspond correctly to the behalves of viscoelasticity; however, when using large time periods, the lack of adequate convergence in the truncated infinite series leads to an unnatural increase in the response of the viscoelastic system. Consequently, when implementing generalized hypergeometric or Mittag-Leffler based models in realistic simulation environments, researchers should carefully select criteria for their choice of truncation limits, time horizon, and numerical stabilization methodologies.

Figure 5 depicts the displacement responses for different fractional viscoelastic damping systems with α=0.6, 0.7, 0.8, and 0.9 fractional orders. The above displacement responses demonstrate the significant impact of the fractional damping orders on damping decay. When α=\alpha\alpha α tends to 1.0, it depicts the classic decay in a memoryless system. When α\alpha\alpha decreases, it represents low damping and memory effects. These properties are pronounced in viscoelastic substances and fractional damping systems. In specific, it can be observed that for α=0.6, there exists a gradual decay in the displacement graph, which signifies material persistence in displacement over a greater period of time characterized by a high level of hereditary behavior. On the other hand, taking larger values of α, there exists a pronounced decay of displacement towards zero, which signifies a high level of material memory and greater efficiency in material damping behavior characterized by a smooth transition from fractional to classic behavior in material response patterns.

Table 1

Fractional Order (\alpha)	Type of Response	Damping Speed	Memory Effect	Physical Interpretation
0.6	Strongly fractional	Very slow decay	Very high	Long memory material – polymer gels, soft tissues
0.7	Moderately fractional	Slow decay	High	Structural dampers, smart materials
0.8	Weakly fractional	Moderate decay	Medium	Composite materials, vibration isolators
0.9	Near–classical	Fast decay	Low	Light damping systems
1.0	Classical integer–order	Exponential decay	None	Ordinary spring–damper systems

6. RESULTS
The obtained results illustrate the importance of the fractional order (α) on the dynamics of the viscoelastic and electrical systems described by fractional-order differential equations. MATLAB simulations using the Mittag-Leffler series reveal that as the value of (α) is reduced from 1.0 to 0.6, there is a progressive slowness in the decay and increased memory effects on the dynamics of the system. These observations confirm the efficacy of the fractional models to capture the uniqueness of hereditary properties which could not be identified using the classical models of integer-order differential equations. For, it reduces to the standard exponential function, which justifies the analytical structure. For, it leads to a nonexponential decay mechanism that characterizes the memory and nonlocality of viscoelastic media and fractional electricity components.

The smooth interpolation for system behavior from classic to fractional systems with regards to confirms its validity in the generalized hypergeometric expression. These results further illustrate that very tiny values for (\alpha) can result in a lack of stability or in the slowing down of the stabilization process, underlining the significance and importance of using proper fractional coefficients in engineering applications. The results further verify the validity and significance of using generalized hypergeometric solutions for fractional-order systems.

7. CONCLUSION
This article introduced a general analytical method for the solution of FODEs arising in engineering systems in terms of generalized hypergeometric functions. Caputo fractional derivatives were used to properly model nonlocal and memory-dependent features associated with actual engineering processes. First, the solutions were represented by means of Mittag-Leffler functions, and then using generalized hypergeometric functions, a unified and powerful representation of fractional dynamics was obtained. The efficiency of the proposed approach has been demonstrated through practical examples such as the fractional RC circuits and the systems with viscoelastic damping. The results showed, quite clearly, that the fractional order \\alpha was critical to system behavior. Smaller values of (\\alpha) resulted in slower decay along with a stronger memory effect; (α = 1) resulted in classical exponential responses.

Simulation using MATLAB, which considers Mittag-Leffler series expansions, verifies the analytical solution and demonstrates how fractional parameters relate to the stability of the system and transient response. The generalized hypergeometric form has several benefits for application in comparison to numerical models, including the allowance for closed-form solutions, easier interpretability, and the ability to analyze sensitivity and control parameters.

This paper, therefore, helps in closing the gap between fractional calculus models and their application in engineering. In the future, it would be worthy of research to extend this work on nonlinear, coupled, and multi-dimensional fractional systems and real-time control applications, which would expand the prominence of hypergeometric functions in fractional modeling.

References

[1] S. Jain, C. Cattani, and P. Agarwal, "Fractional hypergeometric functions," Symmetry, vol. 14, no. 4, Art. no. 714, Apr. 2022.
[2] P. Agarwal, F. Qi, M. Chand, and G. Singh, "Some fractional differential equations involving generalized hypergeometric functions," J. Appl. Anal., vol. 25, no. 1, pp. 1–15, May 2019.
[3] M. Z. Alqarni and M. Abdalla, "Novel kinds of fractional λ–kinetic equations involving the generalized degenerate hypergeometric functions," Mathematics, vol. 11, no. 19, Art. no. 4217, Oct. 2023.
[4] G. Rahman, K. S. Nisar, F. Jarad, and T. Abdeljawad, "Multivariate Mittag–Leffler functions and related fractional integral operators," AIMS Mathematics, vol. 8, no. 6, pp. 13487–13506, 2023.
[5] S. K. Dhaneliya and M. Sharma, "Fractional calculus approach in RLC circuit using hypergeometric series," Turkish J. Comput. Math. Educ., vol. 11, no. 3, pp. 1600–1612, 2021.
[6] J. Restrepo, A. Kılıçman, and P. Agarwal, "Weighted hypergeometric functions and fractional derivatives," Adv. Differ. Equ., vol. 2017, Art. no. 105, Apr. 2017.
[7] R. Yilmazer, H. G. Baskonus, and T. Mekkaoui, "Particular solutions of the confluent hypergeometric differential equation using fractional calculus," Entropy, vol. 18, no. 2, Art. no. 49, Feb. 2016.
[8] H. M. Srivastava, D. Kumar, and J. Singh, "An efficient analytical technique for fractional–order differential equations using generalized Mittag–Leffler functions," J. Math. Anal. Appl., vol. 442, no. 2, pp. 1076–1092, 2016.
[9] xI. Podlubny, Fractional Differential Equations. San Diego, CA, USA: Academic Press, 1999.
[10] R. Gorenflo, A. A. Kilbas, F. Mainardi, and S. Rogosin, Mittag–Leffler Functions, Related Topics and Applications. Berlin, Germany: Springer, 2014.
[11] xA. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, Theory and Applications of Fractional Differential Equations, 2nd ed. Amsterdam, Netherlands: Elsevier, 2006.
[12] F. Mainardi, Fractional Calculus and Waves in Linear Viscoelasticity: An Introduction to Mathematical Models. London, UK: Imperial College Press, 2010.
[13] M. Caputo and M. Fabrizio, "A new definition of fractional derivative without singular kernel," Prog. Fract. Differ. Appl., vol. 1, no. 2, pp. 1–13, 2015.
[14] M. Saigo and M. Maeda, "Fractional calculus and hypergeometric functions," Japan J. Appl. Math., vol. 8, pp. 111–125, 1991.
[15] T. R. Prabhakar, "A singular integral equation with a generalized Mittag–Leffler function in the kernel," Yokohama Math. J., vol. 19, pp. 7–15, 1971.
[16] S. G. Samko, A. A. Kilbas, and O. I. Marichev, Fractional Integrals and Derivatives: Theory and Applications. London, UK: Gordon and Breach Science Publishers, 1993.
[17] V. E. Tarasov, Fractional Dynamics: Applications of Fractional Calculus to Dynamics of Particles, Fields and Media. Berlin, Germany: Springer, 2011.
[18] C. Li and F. Zeng, Numerical Methods for Fractional Calculus. Boca Raton, FL, USA: CRC Press, 2015.
[19] R. Hilfer, Applications of Fractional Calculus in Physics. Singapore: World Scientific, 2000.
[20] S. G. Kiryakova, Generalized Fractional Calculus and Applications. Sofia, Bulgaria: Pensoft, 1994

ISSN 1584 – 2665 (printed version); ISSN 2601 – 2332 (online); ISSN-L 1584 – 2665