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STATE FEEDBACK CONTROL OF A ROBOTIC ARM

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Abstract: In control engineering, a state space representation is a mathematical model of a physical system as a set of input, output and state variables related by first-order differential equations. To abstract from the number of inputs, outputs and states, the variables are expressed as vectors. Additionally, if the dynamical systems linear and time invariant, the differential and algebraic equations may be written in matrix form. The state space representation (also known as the "time-domain approach") provides a convenient and compact way to model and analyze systems with multiple inputs and outputs. With P inputs and Q outputs, we would otherwise have to write down Q x P Laplace transforms to encode all the information about a system. Unlike the frequency domain approach, the use of the state space representation is not limited to systems with linear components and zero initial conditions. "State space" refers to the space whose axes are the state variables. The state of the system can be represented as a vector within that space.
Keywords: control engineering, space representation, mathematical model, differential equations

1. INTRODUCTION

The representation of the state space is a representation in the time domain. Therefore, the system's analysis calls for defined performance indicators in this domain: stability, overshoot, rising time etc. In addition, there are two performance indicators: controllability and observability whose inputs are needed to characterize this representation. Furthermore, we can mention the new possibilities to describe the second order systems, using the state space trajectories and energy curves, which lead to a better understanding of these systems; by using the dominant pole approximation we can determine the systems of higher order.

The matrix form of this representation allows the implementation of numerical algorithms to determine the indices that are mentioned in the case of systems of higher order, which can ultimately lead to a higher accuracy synthesis of the controller.

The representation of the dynamical model of the system in state space is a relation of this type: input - state - output. For linear and continuous systems, this relation is represented by a matrix and can be illustrated as a block diagram (figure 1).

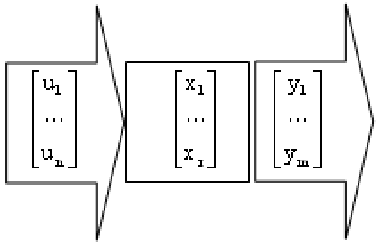


Figure 1. the block diagram

The model described has the following analytical form (1)

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases} \tag{1}$$

in which: x is the state vector of the system (1xr); y is the vector of the output states (1xm); u is the input vector of the control states (input) (1xn); A is the inertia matrix of dimension (rxr) ; B is the command matrix of dimension (rxn) ; C is the output matrix of dimension (mxr); D is the transfer matrix of dimension (mxn) ;

This means the following:

$$\begin{cases} \begin{bmatrix} \dot{x}_1 \\ \cdots \\ \dot{x}_r \end{bmatrix} = \begin{bmatrix} a_{11} & \cdots & a_{1r} \\ \cdots & \cdots & \cdots \\ a_{r1} & \cdots & a_{rr} \end{bmatrix} \begin{bmatrix} x_1 \\ \cdots \\ x_r \end{bmatrix} + \begin{bmatrix} b_{11} & \cdots & b_{1n} \\ \cdots & \cdots & \cdots \\ b_{r1} & \cdots & a_{rn} \end{bmatrix} \begin{bmatrix} u_1 \\ \cdots \\ u_n \end{bmatrix} \\ \begin{bmatrix} y_1 \\ \cdots \\ y_m \end{bmatrix} = \begin{bmatrix} c_{11} & \cdots & c_{1r} \\ \cdots & \cdots & \cdots \\ c_{m1} & \cdots & c_{mr} \end{bmatrix} \begin{bmatrix} x_1 \\ \cdots \\ x_r \end{bmatrix} + \begin{bmatrix} d_{11} & \cdots & d_{1n} \\ \cdots & \cdots & \cdots \\ d_{m1} & \cdots & d_{mn} \end{bmatrix} \begin{bmatrix} u_1 \\ \cdots \\ u_n \end{bmatrix} \end{cases} \tag{2}$$

The state of a system represents the minimum set of variables $x_1, x_2, \ldots, x_r$, the knowledge of which (at moment $t = t_0$) together with the input signal into the system completely determines the behavior of the system at any given time $t \geq t_0$.

The state variables store all the past information of the phenomena ($t < t_0$) that is needed to identify the future evolution of the system. To determine this (minimum) number, the following observation is made: the dynamic analysis for $t = t_0$ requires solving some differential equations, which implies the definition of the initial conditions of the system that stores the previous behavior of the system ($t < t_0$). Consequently, the number of state variables is equal to the number of initial conditions required. Moreover, due to the fact that for the simulation of the integral element systems they are used as memory elements, the outputs of these can be a set of state variables. Thus, the number of state variables becomes equal to the number of integral elements used in the simulation.

The state vector of a system is the vector whose components are state variables $x = [x_1, x_2, \ldots, x_r]^T$.

It is important to point out the fact that the state variables can have different models and can belong or not to the quantifiable physical measurements.

The state space is the r-dimensional space described with the help of the reference axes: $x_1, x_2, \ldots, x_r$; the state of the system, at a certain moment represents a point space with the coordinates $x_1, x_2, \ldots, x_r$. The evolution of the system is described by means of a trajectory referenced by this coordinate system.

The methods used for system's analysis and synthesis that uses the state space representation represents a modern method of investigation. Their success is due to the following:

- ✓ the use of state space representation implies matrix computation which are easily implemented in computer programs;
- ✓ the representation allows a uniform treatment for both SISO and MIMO systems types. In both cases the system is represented by two matrix equations (which differ in terms of size);
- ✓ using this representation allows rewriting models with high order partial differential equations and their transformation into ordinary differential equations (matrix) of first degree;
- ✓ the state space representation has led to methods that allows unified treatment for continuous and discrete systems, linear and nonlinear. [3]

2. THE SYNTHESIS OF A CONTROL SYSTEM FOR A RR ROBOT

The robot that will be controlled has the following structural (mechanical) structure (figure 2)

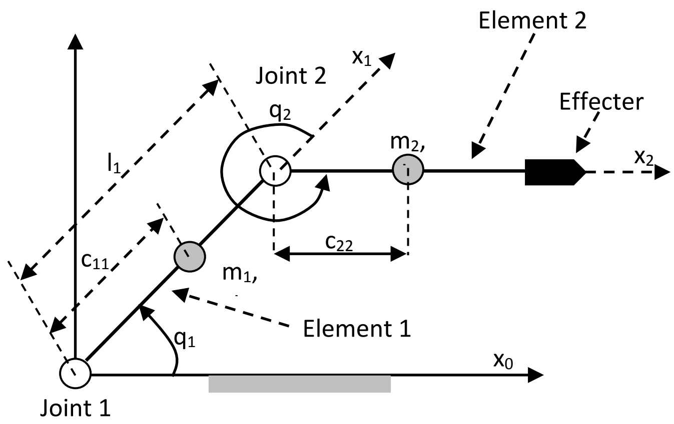


Figure 2. The stucture

$$\begin{bmatrix}\tau_1\\ \tau_2\end{bmatrix}=\begin{bmatrix} m_1c_{11}^2+m_2\left(l_1^2+c_{22}^2\right)+2m_2l_1c_{22}cq_2+{}^1J_{zz}+{}^2J_{zz} & m_2c_{22}^2+m_2l_1c_{22}cq_2+{}^2J_{zz}\\ m_2c_{22}^2+m_2l_1c_{22}cq_2+{}^2J_{zz} & m_2c_{22}^2+{}^2J_{zz}\end{bmatrix}\cdot \begin{bmatrix}\ddot q_1\\ \ddot q_2\end{bmatrix}+\begin{bmatrix}-m_2l_1c_{22}cq_2\dot q_2^2-2m_2l_1c_{22}sq_2\dot q_1\dot q_2\\ m_2l_1c_{22}sq_2\dot q_1^2\end{bmatrix} \quad (3)$$

The dynamic model of the structure is as follows (3) in which:

- ✓ $m_{1/2}$ are the masses of the elements;
- ✓ ${}^{1/2}J_{zz}$ are the inertia moments of the elements related to the z-axis;
- ✓ $l_{1/2}$ are the lengths of the elements;
- ✓ $C_{11/22}$ are the positions of the centers of mass;
- ✓ $bq_{1/2}$ are the angular positions of the joints;

- ✓ sq_1 means $\sin(q_1)$; $cq_2=\cos(q_2)$;
- ✓ $\tau_{1/2}$ are the momentums of the joints.

2.1. Mechanical transmission

The mechanical transmission is mathematical modeled with the following equations (4):

$$\begin{cases} q_{m1/2} = i_{1/2} q_{1/2} \\ T_{1/2} = \dfrac{1}{i_{1/2}} \tau_{1/2} \end{cases} \tag{4}$$

in which: $q_{1/2}$ is the angular position of the joints; $q_{m1/2\ 2}$ is the angular position of the motor shaft 1/2; $\tau_{1/2}$ is the momentum of the two joints, $T_{1/2}$ is the momentum of the two motors; $I_{1/2}$ is the transmission ratio for each motor separately.

The DC motor (it will be used the same type of motor for both joints) is an electro-mechanical system (see figure 3).

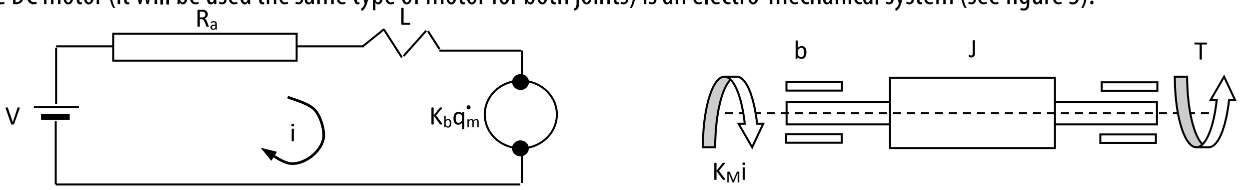


Figure 3. The electro-mechanical system

$$V_{1/2} = R_a i_{1/2} + L\frac{di_{1/2}}{dt} + E_{1/2} \tag{5}$$

$$J q_{m1/2} = K_M i_{1/2} - b\dot{q}_m - T_{1/2} \tag{6}$$

where: V is the supply voltage of the motor; T is the resistive torque; R_a is the winding resistance; L is the winding inductance; i is the winding current; J is the inertia momentum (armature + shaft); $K_M i$ is the driving torque: K_M is the coefficient of the motor torque; b is the coefficient of the viscous friction; q_m is the armature's position; E is the e.m.f. force given by the relation: $E = K_b \dot{q}_m$, in which: K_b is the e.m.f. constant; $\dot{q}_m$ is the rotor speed;

The model's parameters are given in Table 1.

Table 1. The model's parameters

System	Parameter	1	2
Mechanical structure	m [kg]	0.7	0.5
	J [Nms^2/rad]	1.8e-3	0.78e-3
	l [m]	0.5	0.35
	c [m]	0.25	0.125
Transmission	i	150	150
D.C. Motor	J [Nms^2/rad]	233e-6	
	L [H]	0.5	
	Ra [Ω]	0.8	
	b [Nms/rad]	0.1	
	K_M [Nm/amp]	176e-3	
	K_b [Vs/rad]	0.105	

2.2. The analysis of the dynamical model

The analysis of the achieved dynamical model begins with the structural analysis of the RR robot system. Figure 4 shows a block diagram of this system.

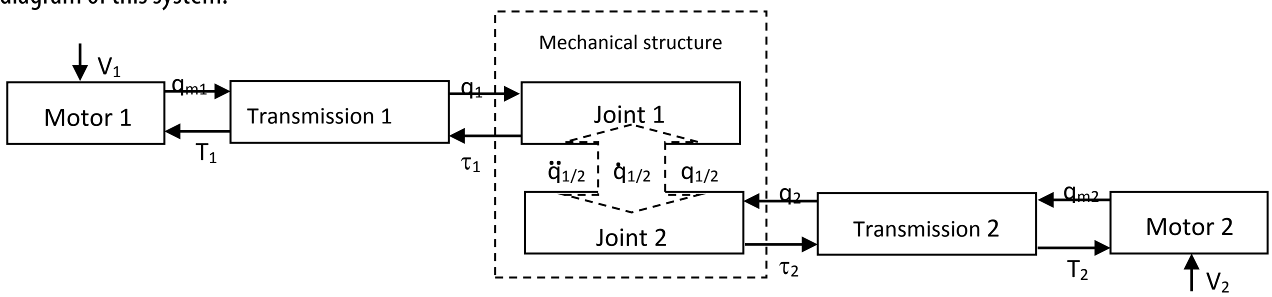


Figure 4. The block diagram of this system

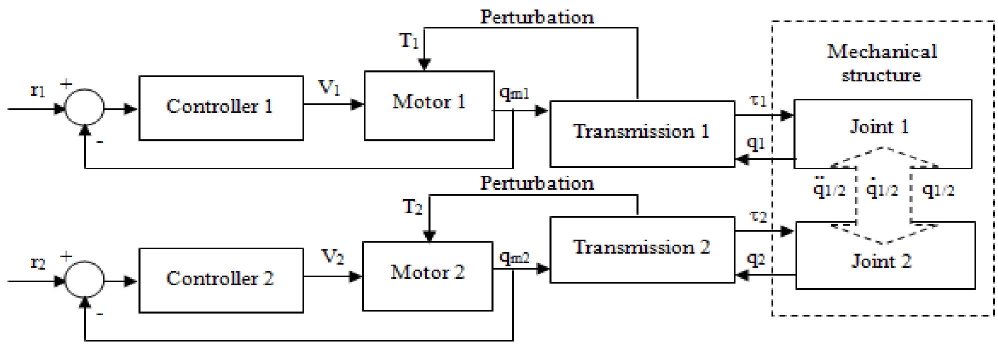


Figure 5. The structure of diagram 4

It can be noted that the connections of the subsystems and the variables transformation.

Because the reference variable controlled in this system is the angle of the motor shaft (referred as $r_{1/2}$ or $qd_{1/2}$), the structure of diagram 4 can be represented as in figure 5:

The problem is reduced to the position control of the two D.C. motors. The system is perturbed with a torque $T_{1/2}$. These momentums are calculated using the equations (3) and (4). We use the Laplace operator on the equations (5) and (6) :

$$Q_m(s)=G_V(s)\cdot V(s)+G_T(s)\cdot T(s) \tag{7}$$

in which: $Q_m(s)$ is the Laplace transform of the value q_m; V(s) is the Laplace transform of the value V; T(s) is the Laplace transform of the value T; $G_V(s)$ is the transfer function:

$$G_V(s)=\frac{K_M}{LJs^3+(JRa+bL)s^2+(bRa+K_bK_M)s} \tag{8}$$

$G_T(s)$ is the transfer function:

$$G_V(s)=\frac{Ls+Ra}{LJs^3+(JRa+bL)s^2+(bRa+K_bK_M)s} \tag{9}$$

If we replace the variables in the transfer functions (G_V and G_T) with the values from table 1 we obtain the following system:

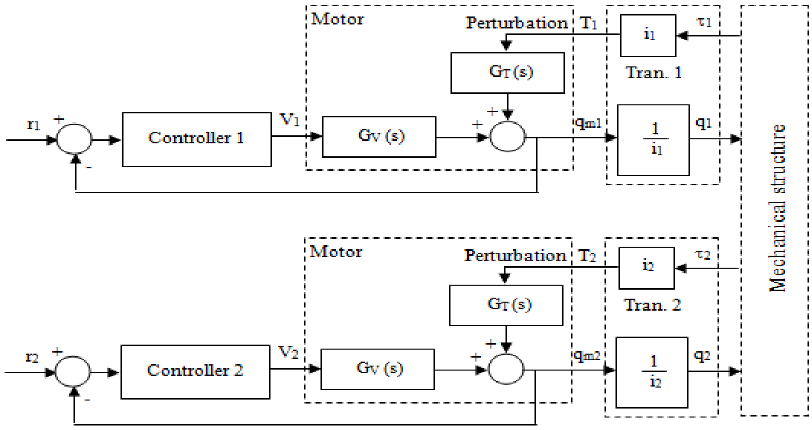


Figure 6. The new system

$$\begin{bmatrix}\tau_1\\ \tau_2\end{bmatrix}=\begin{bmatrix}0.1791+0.0625cq_2 & 0.0086+0.0625cq_2\\ 0.00625+0.0625cq_2 & 0.0633\end{bmatrix}\cdot\begin{bmatrix}\ddot{q}_1\\ \ddot{q}_2\end{bmatrix}+\begin{bmatrix}-0.0313sq_2\dot{q}_2^2-0.0625sq_2\dot{q}_1\dot{q}_2\\ 0.0.0313sq_2\dot{q}_1^2\end{bmatrix} \tag{10}$$

$$G_V(s)=\frac{1511}{s^3+430.8s^2+845.3s} \tag{11}$$

$$G_T(s)=\frac{4291.84s+6866.95}{s^3+430.8s^2+845.3s} \tag{12}$$

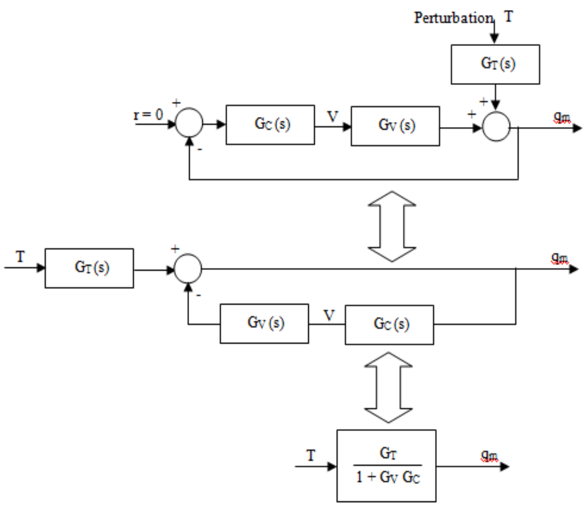


Figure 7. The adopted control strategy

According to the control strategy adopted in this case (fig. 7) the controlled system has the transfer function $G_V(s)$. One may have the following observations, regarding the equation 11:

- ✓ because it has a pole in the origin, the closed loop (undisturbed) system leads to zero stationary error. This can be stated as follows: it is sufficient to use a proportional controller for obtaining a zero stationary error (for step signal);

Several aspects can be noticed:

- ✓ if we use only a P controller the error is not eliminated

$$q_m(\infty)=\frac{4.54}{K} \tag{14}$$

where: K is the proportional coefficient of the controller

In order to eliminate the perturbation ($q_m(\infty)=0$) it is necessary to insert a new controller with a pole in the origin. Consequently, the poles of the transfer function of the plant are:

$$\begin{cases} p_1 = 0 \\ p_2 = -1.9712 \\ p_3 = -428.8288 \end{cases} \tag{15}$$

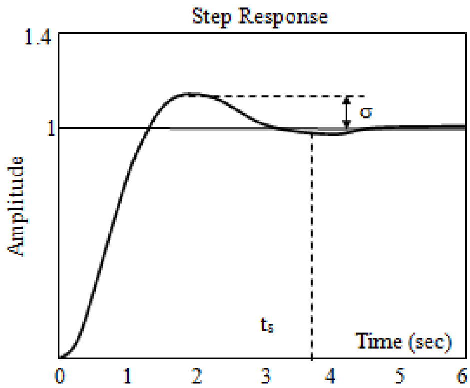


Figure 8. The unperturbed closed loop system

✓ the step response of the unperturbed closed loop system is presented in figure 8:

The overshoot is limited to $\sigma \cong 20\%$ and the settling time $t_s \cong 4$ s. However, in case of industrial robots the imposed performance for the controller needs to be: $\sigma < 5\%$, $t_s < 1$ s.

The disturbance $T_{1/2}$ introduced by the resistive (load) torques ($\tau_{1/2}$) has the property to be boundary limited.

Following observations resulted from this

✓ The control strategy consist in asserting the D.C. motor to the fixed part of the plant and forcing the rest of the subsystems to act as a disturbance generator.

✓ according to the mathematical formulations of this systems the disturbances which occur are boundary limited.

2.3. Controller synthesis

The state vector has the following structure:

$$x = [q \quad \dot{q} \quad i] \tag{16}$$

where: q is the position angle (degrees); q is the angular speed of the rotor shaft; i is the current supplied to the motor

Consequently, the new form of the system becomes:

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -\frac{b}{J} & \frac{K_M}{J} \\ 0 & -\frac{K_b}{L} & -\frac{R_a}{L} \end{bmatrix} \cdot x + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{L} \end{bmatrix} \cdot u + \begin{bmatrix} 0 \\ -\frac{1}{J} \\ 0 \end{bmatrix} \cdot T \tag{17}$$

$$y = [1 \quad 0 \quad 0] \cdot x \tag{18}$$

That which, by replacing of the values from table 1 results:

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -429.1845 & 755.3648 \\ 0 & -0.21 & -1.6 \end{bmatrix} \cdot x + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \cdot u + \begin{bmatrix} 0 \\ -4291.8 \\ 0 \end{bmatrix} \cdot T$$
$$y = [1 \quad 0 \quad 0] \cdot x \tag{19}$$

according to the principle of superposition of effects, the model of the unperturbed system is:

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -429.1845 & 755.3648 \\ 0 & -0.21 & -1.6 \end{bmatrix} \cdot x + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \cdot u$$
$$y = [1 \quad 0 \quad 0] \cdot x \tag{20}$$

and the perturbation's effect over the motor (the model of the perturbation) is:

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -429.1845 & 755.3648 \\ 0 & -0.21 & -1.6 \end{bmatrix} \cdot x + \begin{bmatrix} 0 \\ -4291.8 \\ 0 \end{bmatrix} \cdot T$$
$$y = [1 \quad 0 \quad 0] \cdot x \tag{21}$$

By applying the synthesis algorithm, the following steps are needed:

Step 1: Implementing the problems' initial data:

$$A=\begin{bmatrix}0 & 1 & 0\\ 0 & -429.1854 & 755.3648\\ 0 & -0.21 & -1.6\end{bmatrix};\ B=\begin{bmatrix}0\\0\\2\end{bmatrix};\ C=\begin{bmatrix}1\\0\\0\end{bmatrix}^T;\ \begin{cases}\gamma_1=-8\\ \gamma_2=-8\\ \gamma_3=-20\\ \gamma_4=-20\end{cases} \tag{22}$$

$$\hat{A}=\begin{bmatrix}0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & -845.3 & -430.8 & 0\\ -1511 & 0 & 0 & 0\end{bmatrix};\ \hat{B}=\begin{bmatrix}0\\0\\1\\0\end{bmatrix};\ \hat{K}=\begin{bmatrix}k_1 & k_2 & k_3 & -k_1\end{bmatrix} \tag{23}$$

Step 2: Controllability check:

$$\mathrm{rang}\begin{bmatrix}\hat{B} & \hat{A}\hat{B} & \hat{A}^2\hat{B} & \hat{A}^3\hat{B}\end{bmatrix}=4: \tag{24}$$

the system is controllable;

Step 3: The calculus of the characteristic polynomial:

$$\Theta(s)=s^4+56s^3+110s^2+8960s+25600 \tag{25}$$

Step 4:.The determination of the agreement matrix parameters:

$$\hat{K}=\begin{bmatrix}0 & 0 & 0 & 1\end{bmatrix}\cdot\begin{bmatrix}\hat{B} & \hat{A}\hat{B} & \hat{A}^2\hat{B} & \hat{A}^3\hat{B}\end{bmatrix}^{-1}\cdot\Theta(\hat{A})=\begin{bmatrix}5.9309 & 106.6441 & -187.3923 & -16.94\end{bmatrix} \tag{26}$$

Step 5: The simulation result is achieved using the scheme in Figure 9 (Simulink). The artifice that allowed the use of this scheme was the transformation of the output matrix in:

$$C=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} \tag{27}$$

The K blocks were also used in the scheme: K_i: 16.94 - gain;

- matrix's gain:

$$K=\begin{bmatrix}5.9309 & 106.6441 & -187.3923\end{bmatrix};$$

The simulation results: the step response of the system and the perturbation's annulations are represented in the Table 2.

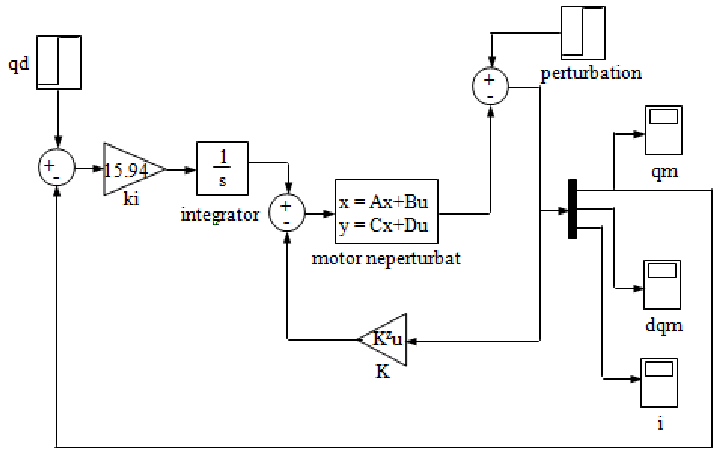
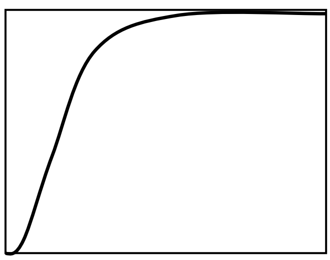
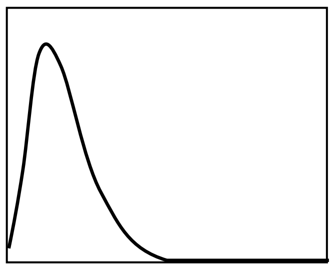


Figure 9. The simulation result

Table 2. The system and the perturbation's annulations

Step response	Perturbation's annulations
	
qd is the unit step input signal; the perturbation is zero. The response acquired is a critical damped signal, which is compatible with the objectives of the control system. In addition, the settling time is reduced at 1 sec.	qd is nil; the perturbation is a step signal. Finally, the perturbation is totally annihilated.

The result obtained allows the simulation of the entire robot. Thus, a block diagram can be implemented in Simulink (figure 10); this diagram must also simulate the effect of the perturbations (the dynamic of the mechanical structure – (26)) over the two controlled D.C. motors.

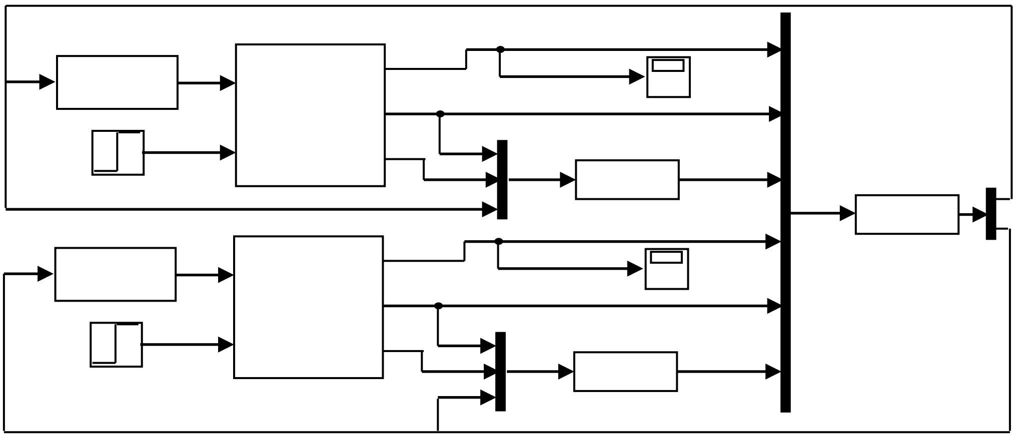
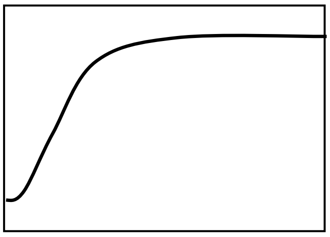
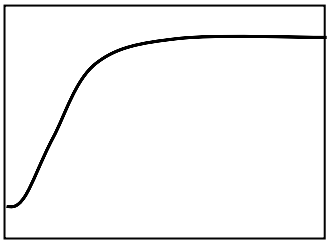
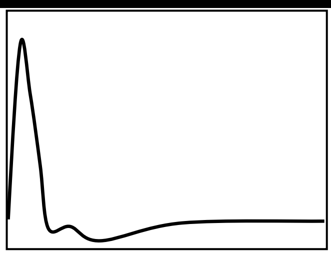
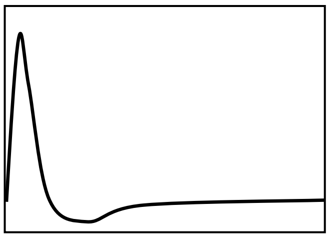


Figure 10. Block diagram implemented in Simulink

The scheme for the remote controlled robot, presented in figure 10, contains the following blocks:

- ✓ $qd_{1/2}$ are unitary step signal generators and they represent the desired position angles in the robot's joints;
- ✓ The disturbances$_{1/2}$ are the blocks which use the representation in state space and which transfer in Simulink the disturbance model (diagram 7). The blocks are equivalent but have different outputs T_1 and T_2;
- ✓ The structure is a block, which results from a MATLAB function and which allows the determination of the two moments $T_{1/2.}$ The function is the transposing in program of the equations (3) and (4). That is, it is the solution for the reverse dynamic model of the structure, in conjunction with the cinematic model of the movement transmission. The input data of the function are: q_{m1} $\dot{q}_{m1}$ $\ddot{q}_{m1}$;q_{m2} $\dot{q}_{m2}$ $\ddot{q}_{m2}$; and the output ones are: T_1 and T_2;
- ✓ $ddq_{1/2}$ are two blocks which calculate the accelerations from the robot's joints. In the case of this control strategy through the adoption of the state vector (diagram 4) it has been created the possibility to implement the controller practically. This fact has led to the necessity of the indirect calculation of the accelerations from the robot's joints. It is important to mention the fact that this aspect occurs only in case of system simulation and the fact that the mentioned block is based on equation (6)
- ✓ $q_{m1/2}$ are the block which allow the display of the variables which occur in the robot's joints and which are systematized in table 3.

Table 3. The variables which occur in the robot's joints

Motor 1	Motor 2
The rotating angle of the motor shaft	
	
The desired angle of the joint was modeled using a unit step signal. More specifically, the robot was required to rotate each coupling with an angle of a radian. In both cases it can be observed that the objectives sets are achieved: establishing an operating mode for obtaining a critically amortization regime and a stabilization time smaller than a second.	
The supply voltage of the two motors	
	
The low voltage of the two DC motors is due to the high ratio of transmission used for the transmission of the movement from the motor to the structure.	

The result acquired allows the achievement of the performance indices imposed (censorious behavior damped with the growth time shorter that one second). However, for a greater security in the validity of the proposed solution, there are needed some simulations of the controlled system and in the frequency domain.

3. CONCLUSIONS

The present paper approaches the design problem of the control system of a robotic arm using the space state. The solution suggested can be easily implemented in practice and allows the acquisition of some very good driving performances, making use of the whole model of the system (motor + transmission + structure).

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